Homework Set 2

Read sections 2.5, 2.7, 3.1, and 8.1.

Section 2.5

1. (BH) The cruise ship *Norwegian Joy* leaves Los Angeles with 3200 passengers, 5 of whom have the flu. By the end of the next day, 10 people have the flu.
   (a) If the number of people with the flu $N(t)$ spreads according to the exponential growth model, calculate $N(t)$.
   (b) How many people will have caught the flu by the end of the week-long cruise?

2. Suppose that due to weather and other habitat-related conditions, the growth rate actually fluctuates over the course of a year. This can be modeled by

$$\dot{N} = (r \cos kt)N, \qquad N(0) = 1,$$

   where $r > 0$ and k are constants.
   (a) (BH) Find the solution to this problem.
   (b) (MP) Graph your solution for $r = 2$, $k = \pi$, $t \in [0, 10]$.

3. (BH) A population $P(t)$ satisfies the following *Gompertz model*:

$$\dot{P} = rP \log\left(\frac{K}{P}\right), \qquad P(0) = 1, \tag{2.1}$$

   where $r > 0$ and $K > 0$ are constants.
   (a) Find the solution to this problem.
   (b) How does the solution behave as $t \to \infty$?
   (c) Are there any values of K for which the population remains constant?

Sections 2.7/8.1

4. Consider the equation

$$\frac{dy}{dt} = \sin(100ty), \qquad y(0) = 1.$$

   (a) (BH) Explain why $y(1) < 2$. (*Hint: What's the largest $\dot{y}$ can be?*)
   (b) (BH) Write d^2y/dt^2 as a function of y and t.
   (c) (BH) Let $t \in [0,1]$. Explain why if we take $\Delta t \approx 1/150$, the error term in Euler's method is roughly the same size as the Euler step itself. (*Hint: Use the worst-case estimates for all the terms.*) This then enforces an upper bound on Δt.
   (d) (MP) Plot the solution for $t \in [0,1]$ using the default Mathematica solver, Euler's method with $\Delta t = 1/50$, and Euler's method with $\Delta t = 1/450$. Compare your results.

5. (MP) Consider the equation

$$\dot{y} = t^2(y^2 - 1), \qquad y(0) = 0. \tag{2.2}$$

   (a) Calculate the solution.
   (b) Let $E(t;\Delta t)$ be the solution of (2.2) calculated in Mathematica using Euler's method with stepsize Δt. Calculate and store the ordered pairs

$$[\Delta t, |E(1;\Delta t) - y(1)|], \qquad \Delta t = \frac{1}{N}, \quad N = 10, 11, \ldots, 50,$$

   where $y(1)$ is the exact solution at $t = 1$.
   (c) Plot your answer to (b) and verify that the points lie on a line. Thus we know that the error is proportional to Δt, as predicted by the theory.

Section 3.1

6. (BH) Consider the differential equation

$$2\ddot{y} + 3\dot{y} + y = 0.$$

   (a) Find the general solution. Describe the long-time behavior.
   (b) Calculate the specific solution for $y(0) = 3$, $\dot{y}(0) = -4$.

7. (BH) Write down *all* equations of the form $a\ddot{y} + b\dot{y} + cy = 0$ such that the solution y approaches a multiple of e^{2t} as $t \to \infty$.

8. (BH) Consider the following system of coupled first-order ODEs:

$$\dot{x} = 4x + 2y, \tag{2.3a}$$
$$\dot{y} = 3x - y. \tag{2.3b}$$

   (a) Eliminate x from the system to obtain a second-order ODE for y.
   (b) Show that the general solution for y is

$$y(t) = c_1 e^{-2t} + c_2 e^{5t},$$

   and find the corresponding general solution for x.

9. (MP) Consider the differential equation

$$\ddot{y} + 3\dot{y} - 12y = 0, \quad y(0) = y_0, \quad \dot{y}(0) = 0. \tag{2.4}$$

   (a) Using `DSolve`, calculate the solution of (2.4).
   (b) Plot your results for $y_0 = -2, -1, 0, 1, 2$, and $t \in [0, 1/2]$.

10. Consider the differential equation

$$\ddot{y} + 2\dot{y} + (1 - \epsilon)y = 0, \quad y(0) = 0, \quad \dot{y}(0) = 1 - \epsilon. \tag{2.5}$$

   (a) (MP) Use `NDSolve` to plot the solution to (2.5) for $\epsilon = -0.2, -0.1, 0, 0.1$, and 0.2, and $t \in [0, 2]$.
   (b) (BH) Is your graph for $\epsilon = 0$ consistent with what you would expect from substituting in $e^{\lambda t}$? (We will resolve this paradox in a later section.)