MATH 302-010 Ordinary Differential Equations
Prof. D. A. Edwards Due: Sept. 13, 2024

Use of Mathematica

Each problem has a code that indicates how you are to use Mathematica on your submitted work. (You may always use Mathematica to check your work.) The codes are:

BH: (by hand) anything submitted should be done by hand and should show all steps.

MI: (Mathematica integral) you may use Mathematica to calculate certain integrals, but the rest of the problem should be done by hand. Provide printouts of any Mathematica integral calculations.

MP: (Mathematica printout) use Mathematica as appropriate to get the answer and give me a printout of your worksheet.

Homework Set 1

Read sections 1.1, 2.1, 2.2, and 2.4.

Section 1.1

1. (BH) Consider the differential equation

$$\dot{y} + y^2 = 9. \tag{1.1}$$

   (a) Find any equilibrium solutions.
   (b) Sketch a direction field for (1.1). Indicate the position of the equilibrium solutions.
   (c) What does your graph tell you will happen to the solution as $t \to \infty$? Be sure to discuss all possible initial conditions.

2. (MP) Consider the differential equation

$$\dot{y} + \sin y = 1.$$

   Construct a graph showing the direction field and any equilibrium solutions in $t \in [0, 2\pi]$, $y \in [-5, 5]$.

Section 2.1

3. Consider the differential equation

$$\dot{y} - 4y = 2e^{-t}, \qquad y(0) = y_0.$$

   (a) (BH) Find the solution for any constant y_0.
   (b) (BH) Describe how the long-time behavior of y varies with y_0. (In other words, does the solution decay, tend to positive or negative infinity, etc.)
   (c) (BH) Find the critical value of y_0 which separates the two types of behaviors.
   (d) (BH) Describe the long-time behavior of y for that specific value of y_0.
   (e) (MP) Using the solution you derived in (a), plot integral curves of $y(t)$ for $t \in [0, 0.5]$ and various y_0. Be sure to include the value of y_0 derived in (c).

4. (BH) Show (by deriving the solution, **NOT** by direct substitution) that the solution to the differential equation

$$t\dot{y} + 3(2t + 1)y = e^{-6t}, \qquad y(1) = 0$$

   is given by

$$y(t) = \frac{e^{-6t}}{3}\left(1 - \frac{1}{t^3}\right).$$

5. (This problem is designed to make you realize that you cannot rely blindly on Mathematica's answers.) Consider the following ODE:

$$\tan\left(\frac{1}{t}\right)\dot{y} + \frac{y}{t^2} = 0.$$

   (a) (BH) Calculate the general form for $y(t)$.
   (b) (MP) Calculate the solution when $y(1/\pi) = 0$ using `DSolve`.
   (c) (MP) Calculate the solution when $y(1/\pi) = 0$ using `NDSolve` and plot it for $t \in [-1, 1]$.
   (d) (BH) Calculate the solution when $y(1/\pi) = 0$. Do your Mathematica answers miss anything?

Section 2.4

6. (BH) exercise 21

7. (BH) Consider the differential equation

$$t^3\dot{y} + 2t^2 y = t^4 + t^5, \qquad y(1) = y_0.$$

   (a) Find the general solution. Where is the solution defined, in general?
   (b) Are there any particular values of y_0 for which the solution is defined everywhere? If so, calculate them. If not, explain why not.

Section 2.2

8. (BH) Consider the equation

$$\dot{y} + y^2 = 0, \qquad y(0) = y_0 < 0.$$

   (a) Write down the solution to the equation.
   (b) How does the interval of existence for the solution depend on y_0?

9. (BH) Show (by deriving the solution, **NOT** by direct substitution) that a solution of the equation

$$y(t^2 - 1)\dot{y} = t(y^2 - 1), \qquad y(0) = 0,$$

   is $y = t$. Are there any others? Explain your answer in light of the existence and uniqueness theorem.

10. Consider the equation

$$\dot{w} = -kt^{\alpha} w^3, \quad w(1) = 1, \tag{1.2}$$

   where $k > 0$ and α are constants.

   (a) (BH) Find the solution of (1.2). Be sure to examine the special case when $\alpha = -1$.
   (b) (MP) Check your answer using Mathematica. Does Mathematica give the solution to every case automatically?
   (c) (BH) Discuss the behavior of the solutions to (1.2) as $t \to \infty$. Remark on the solution for all α.
   (d) (MP) Plot integral curves for $k = 3$, $t \in [1, 5]$, and $\alpha = -2, -1, 0, 1, 2$.