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Measurement of modal power in optical fibers

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1 Introduction

Increasing demands on communications networks to achieve higher carrying capacities, faster transmission rates and better signal reliability require the development of improved testing procedures and the establishment of new communication standards. Over the coming decade, local area networks (LAN's) will be upgraded to operate at bit rates one hundred times faster than current standards. The gigabit ethernet (GBE) will achieve transmission rates of one gigabit per second over multimode optical fibers with high speed laser signal sources. To optimize the signal characteristics of laser pulses in the optical fibers, it is important to understand how dispersion, radiation, tunneling and other effects work to distort the signal. In the workshop, we addressed questions concerning what properties of the input laser pulse could be recovered from measurements of the output signal intensity. In particular, we examined a method to calculate the power sent through each of the propagating linear modes in a multimode optical fiber from a measurement of the near-field intensity $I(r)$. The goal of our work is to give a robust estimate of the modal power distribution which is not overly sensitive to measurement noise, is internally self-consistent, and provides an improvement over existing techniques [2, 6, 11].

Following a brief review of the governing equations appropriate to the geometry for optical fibers, we summarize our study of the modal power problem. We will begin with the forward problem, namely, given a input electric field determine the output intensity function. The inverse problem of what can be learned about the input from only the intensity function will also be addressed.

2 Governing equations

The fundamental equations describing the transmission of light in optical fibers are Maxwell's equation for electromagnetic waves propagating in media. In the absence of free charges or currents, the linear wave equation for the electric field is

$$\mu\epsilon\frac{\partial^2 \mathbf{E}}{\partial t^2} = \nabla^2 \mathbf{E}, \tag{1}$$

where ϵ, μ are the permitivity and permeability of the medium. For the study of digital communications in optical fibers there is a single direction of wave propagation that is of interest. Methods developed for the analysis of transmission modes in waveguides [13] can be applied to fiber optic communications, treating the fiber as a waveguide with a circular cross-section. The description of waves directed along the axis of the fiber can be given in terms of the solution of a scalar wave equation,

$$\frac{n^2(r)}{c_0^2}\frac{\partial^2 E}{\partial t^2} = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial E}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 E}{\partial \theta^2} + \frac{\partial^2 E}{\partial z^2}, \tag{2}$$

written in cylindrical coordinates aligned with the axis of the optical fiber. All of the components of the electric and magnetic fields described by (1) can be expressed in terms of the solution of (2). The important material properties of the optical fiber are specified by the index of refraction, $n = n(r)$, which gives the ratio of the speed of light in vacuum to the speed of light in the fiber, $n = c_0/c \geq 1$ where $c_0 = 1/\sqrt{\mu_0\epsilon_0}$. We examine graded index multimode fibers where the index of refraction is given by

$$n(r) = n_0\sqrt{1-(r/R)^\alpha}, \qquad 0 \leq r < R, \tag{3}$$

for $\alpha \geq 1$, where n_0 is the index of refraction at the center of the fiber ($r = 0$), and R is some radial lengthscale. The actual optical fiber consists of a core region $0 \leq r \leq R_1$ supporting transmission, and a layer of cladding $R_1 < r \leq R_2$ inside a protective jacket. Equation (3) describes the index of refraction in the core region $r < R_1$, where $n_0 \geq n(r) \geq n_1 = n(R_1) > 1$, and the bulk of the propagation of light occurs. In the outer regions $r \gg R_1$, where we assume that $n \sim n_1$, the details of the localized, propagating solutions should be insensitive to properties of the media away from the core of the optical fiber (3).

The general solution of equation (2) can be expressed as a linear combination of modal solutions,

$$E(r,\theta,z,t) = \sum_{\mathbf{i}} a_{\mathbf{i}} E_{\mathbf{i}}(r,\theta,z,t), \tag{4}$$

where the sum over the "multi-index" $\mathbf{i}$ describes all possible modal solutions. Using separation of variables each of these modal solution can be written as products of the form

$$E(r,\theta,z,t) = \psi(r)e^{im\theta}e^{i(kz-\omega t)}, \tag{5}$$

where the radial eigenfunctions $\psi(r)$ satisfy the ordinary differential equation,

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + U(r)\psi = 0, \qquad 0 \leq r < \infty, \tag{6}$$

and the radial potential function is

$$U(r) = \frac{\omega^2 n^2(r)}{c^2} - \frac{m^2}{r^2} - k^2. \tag{7}$$

Equation (6) is a singular boundary value problem for $\psi(r)$, of the same structure that occurs in quantum mechanics problems for the wavefunctions of bound states of electrons in atomic orbitals (as is described in some introductory textbooks on fiber optics [3, 4, 10]). For our fiber optics application, the bound eigenstates, satisfying $\psi(r \to \infty) \to 0$, describe propagating electromagnetic waves that are localized to the core of the fiber. There are also other solutions of (6) that represent solutions with ionization, tunneling or evanescent wave effects that will not contribute to the transmission of signals in the fibers. For an optical fiber of finite radius there are a finite number of discrete eigensolutions; fibers designed to make use of these higher modes (not just the fundamental solution) are called multimode fibers.

A fundamental property of modal solutions of (2) is orthogonality of different modes integrated over all space,

$$\int_0^\infty \int_0^{2\pi} \bar{E}_j E_i r \, dr \, d\theta = 0, \qquad i \neq j. \tag{8}$$

As a consequence, the total power transmitted is given by the sum of the modal power components,

$$\int\int |E|^2 r \, dr \, d\theta = 2\pi \sum_i |a_i|^2 \int_0^\infty \psi_i^2(r) r \, dr, \tag{9}$$

this is a statement of Parseval's theorem. Equation (9) suggests defining an intensity function $I(r)$ as

$$I(r) = \sum_i p_i \psi_i^2(r), \tag{10}$$

where the modal power is given by the square of the modal amplitude, $p_i = |a_i|^2$. We note however, that this intensity function is **not** equal to the square of the amplitude of the electric field,

$$\begin{aligned} |E|^2 &= \sum_i \sum_j a_i \bar{a}_j \psi_i(r) \psi_j(r) e^{i\theta\Delta m} e^{iz\Delta k} e^{-it\Delta\omega} \\ &= \underbrace{\sum_i |a_i|^2 \psi_i^2(r)}_{I(r)} + \underbrace{\sum_{i \neq j} a_i \bar{a}_j \psi_i(r) \psi_j(r) e^{i\theta\Delta m} e^{iz\Delta k} e^{-it\Delta\omega}}_{\text{"cross terms"}}, \end{aligned} \tag{11}$$

where $\Delta b \equiv b_i - b_j$ for $b = k, m, \omega$ above. The cross terms present above will generally not vanish, and in fact allow the electric field amplitude to have (θ, z, t)-dependence that is not present in the mean field intensity given by $I(r)$. As described in [9], measurements of the optical power derived from the Poynting vector yield such modal cross product terms that only vanish when they are integrated over all space,

$$\int_0^\infty \psi_i(r) \psi_j(r) r \, dr \cdot \int_0^{2\pi} e^{i(m_i - m_j)\theta} \, d\theta = 0, \qquad i \neq j. \tag{12}$$

It should be noted that measurement of the intensity is not done instantaneously, and hence it will involve averaging over a time period T,

$$\overline{|E|^2} = I(r) + O\left(\frac{1}{T} \int_0^T e^{it\Delta\omega} \, dt\right) = I(r) + O\left(\frac{1}{T\Delta\omega}\right). \tag{13}$$

Hence it is possible to justify neglecting the influence of the cross terms for long time averages, $T \to \infty$. Weak angular dependence observed in the intensity field, sometimes called "speckle patterns", are also attributable to these cross terms. Questions were raised (and they still remain) about whether $I(r)$ accurately describes what is being measured by intensity meters used in the experiments on real optical fibers. As will be shown, it is also not clear that $I(r)$ is the most convenient quantity to measure in such a transmission test, or that it is a sufficient measurement to provide all the desired information.

We will now illustrate what information can be obtained about the values of the modal power contributions p_i from a measurement of the intensity function $I(r)$ for the specific case of a quadratic profile graded fiber. This will be followed by a discussion of some of limitations and other engineering considerations that enter into this problem, but are not clearly answered or justified within the framework of the linear governing equation (2).

3 Modal power for the quadratic index graded fiber

Taking $\alpha = 2$ in (3) produces what is called the quadratic index profile; this model for the index of refraction has a convenient analytic solution that we make use of. Following the notation used in [10], the equation, (7), for the radial solutions $\psi(r)$ becomes

$$\frac{d^2\psi}{dr^2} + \frac{1}{r}\frac{d\psi}{dr} + \left(\sigma^2 - \frac{m^2}{r^2} - T^2r^2\right)\psi = 0, \qquad 0 \le r < \infty, \tag{14}$$

where

$$\sigma^2 = n_0^2\omega^2\mu_0\epsilon_0 - k^2, \qquad T = n_0\omega\sqrt{\mu_0\epsilon_0}/R. \tag{15}$$

The solutions of this differential equations are eigenmodes,

$$\psi_{\ell,m}(r) = \sqrt{\frac{\ell! T}{(m+\ell)!\pi}}(Tr^2)^{m/2}e^{-Tr^2/2}L_\ell^m(Tr^2), \tag{16}$$

where $L_\ell^m(x)$ are the Laguerre polynomials of degree ℓ and order m, given by the recursion relation

$$L_\ell^m(x) = \tfrac{1}{\ell}\left([2\ell + m - 1 - x]L_{\ell-1}^m(x) - [\ell + m - 1]L_{\ell-2}^m(x)\right), \qquad \ell = 2, 3, \cdots, \qquad m = 0, 1, 2, \cdots, \tag{17}$$

where $L_0^m(x) = 1$, and $L_1^m(x) = 1 + m - x$. The eigenmodes (16) correspond to the eigenvalues for σ,

$$\sigma^2 = 2(2\ell + m + 1)T. \tag{18}$$

Using (18) with (15) yields the dispersion relation for the wavemodes,

$$\omega_{\ell,m}(k) = \frac{2\ell + m + 1 \pm \sqrt{(2\ell + m + 1)^2 + k^2R^2}}{n_0R\sqrt{\mu_0\epsilon_0}}. \tag{19}$$

The overall electric field, for a given k can then be written as

$$E = \sum_{\ell,m} a_{\ell,m}\psi_{\ell,m}(r)e^{im\theta}e^{i(kz-\omega t)}, \tag{20}$$

with the corresponding intensity function

$$I(r) = \sum_{\ell,m} p_{\ell,m}\psi_{\ell,m}^2(r). \tag{21}$$

Different modes having the same value of $(2\ell + m)$ yield solutions with the same phase velocity, $c_p = \omega/k$, and the same group velocity, $c_g = d\omega/dk$, and are said to belong to the same degenerate mode group. Since the group velocity depends on k, this is a dispersive system, so the various modes composing an initial wave pulse will separate as the pulse propagates. However, modes belonging to the same degenerate group propagate together and some studies have simplified (20) as a summation over different mode groups. Some comments on results for degenerate mode groups will be given later. For the preliminary work done during the workshop, another simplifying assumption, that the electric field was axi-symmetric (only modes with $m = 0$ contribute to the solution).

3.1 The direct problem: $E(r) \rightarrow I(r)$

To better understand the relation of the incident electric field to the resulting output intensity function, we carried out some analytical calculations for a few elementary input beam energy distributions including Gaussian, triangular and parabolic beams.

The Laguerre polynomials $L_\ell^m(x)$ for any fixed m, form a complete set and thus can be used as a basis for arbitrary $E(r)$.[1] Considering axially symmetric input fields, we write

$$E(r) = \sum_\ell a_\ell \psi_\ell(r), \tag{22}$$

where $\psi_\ell(r)$ are given by (16). Using the orthogonality of the $\psi_\ell(r)$, we obtain

$$a_\ell = \int_0^\infty E(r)\psi_\ell(r) r\, dr. \tag{23}$$

Now the quantities $I(r)$ and $|E|^2$ defined in (10) and (11) respectively are easily obtained.

We considered several different $E(r)$ and truncated expansion (22) to the first ten modes and then computed (10) and (11). The use of more modes did not qualitatively change the results. In addition to a class of Gaussian beams,

$$E(r) = e^{-(r/b)^2}), \tag{24}$$

we considered the following three different source electric fields; delta pulses (triangular pulses),

$$E_1(r) = \begin{cases} 1 - |r|/b & |r| < b \\ 0 & |r| > b \end{cases} \tag{25}$$

parabolic pulses,

$$E_2(r) = \begin{cases} 1 - (r/b)^2 & r < b \\ 0 & r > b \end{cases} \tag{26}$$

and offset pulses (here corresponding to a toroidal beam)

$$E_3(r) = \begin{cases} 1 - ((r-c)/b)^2 & |r - c| < b \\ 0 & |r - c| > b \end{cases} \tag{27}$$

[1] The model for the potential used for this analytic solution does not support any evanescent modes, hence all solutions are expressible in terms of the bound eigenmodes.

The parameter b is a measure of the width of the input beam in each case.

In all cases, cross terms were eliminated only when the majority of the power was propagated in the fundamental mode, i.e. $|a_\ell| \ll 1$ for all $\ell \geq 1$ and $|a_0| > 1$. In particular for $b = 1.18$ for $E_1(r)$, $a_0 = 6.10$, $a_1 = 0.12$, $a_2 = -0.5$, $a_3 = -0.04$, $a_4 = 0.2$, $a_5 = .2$, $a_6 = 0.1$, $|a_7|$, $|a_8|$, $|a_9| < 0.09$. Thus elimination of cross terms is achieved when the multi-mode fiber actually behaves like a single mode fiber. For none of the three inputs, did cross term cancellation result when more than one mode carried a non-trivial power. In these cases, $|E|^2$ and the proposed $I(r)$ differed dramatically.

We next considered the $I(r)$ generated by $E_1(r)$ and $E_2(r)$ for common values of b. We found that the generated $I(r)$ profiles were qualitatively similar for both inputs. This is not surprising given that the inputs are themselves similar. From the engineering viewpoint however, this similarity may be disadvantageous unless there is a quick and easy way to quantitatively discriminate between these two inputs.

Some work was also done to explore the form of the intensity function resulting from a uniform distribution of power among all degenerate mode groups.

3.2 The inverse problem: $I(r) \to E(r)$

Our goal is now, given a function $I(r)$, to obtain the modal power coefficients $p_{\ell,m}$ in (21),

$$I(r) = \sum_{\ell,m} p_{\ell,m}\psi_{\ell,m}^2(r).$$

and then to comment on the corresponding electric fields (20).

The difficulty in this problem is that (21), unlike (20), is not an orthogonal eigenfunction expansion; inner product integrals of $\psi_{\mathbf{i}}^2(r)$ and $\psi_{\mathbf{j}}^2(r)$ (note the squares) do not vanish for $\mathbf{i} \neq \mathbf{j}$. However, it is possible to make use of an integral identity for the squares of the Laguerre polynomials to construct a procedure to obtain these power coefficients. Standard references on orthogonal polynomials [1, 7] include the integral involving the square of the Laguerre polynomials,

$$\int_0^\infty x^{\nu+1}e^{-\beta x^2}L_n^{\nu/2}(\alpha x^2)^2 J_\nu(xy)\,dx = \frac{y^\nu e^{-y^2/(4\beta)}\Gamma(n+1+\nu/2)}{(2\beta)^{\nu+1}\pi n!}\times \tag{28}$$
$$\sum_{\ell=0}^{n}\frac{(-1)^\ell\Gamma(n-\ell+1/2)\Gamma(\ell+1/2)}{\Gamma(\ell+1+\nu/2)(n-\ell)!}\left(\frac{2\alpha-\beta}{\beta}\right)^{2\ell}L_{2\ell}^{\nu}\left(\tfrac{\alpha}{2\beta(2\alpha-\beta)}y^2\right).$$

Specializing this relation to the wavemodes for the fiber optics problem, it takes the form

$$\int_0^\infty \psi_{\ell,m}^2(r)J_{2m}(rs)r\,dr = \frac{1}{\pi^2}\sum_{j=0}^{\ell}\frac{(-1)^j\Gamma(\ell-j+1/2)}{(\ell-j)!}\tilde{\psi}_{2j,2m}(s) \tag{29}$$

where

$$\tilde{\psi}_{2j,2m}(s) = \frac{\Gamma(j+1/2)}{2^{m+1}(m+j)!}(Us^2)^m e^{-Us^2/2}L_{2j}^{2m}(Us^2), \qquad U = \frac{1}{2T} \tag{30}$$

To take advantage of this analytical result, we must assume that we can separate the intensity into contributions from different azimuthal modes,

$$I(r) = \sum_m I_m(r), \qquad I_m(r) = \sum_\ell p_{\ell,m}\psi^2_{\ell,m}(r). \tag{31}$$

We define the Hankel transform functions as

$$H_m(s) = \int_0^\infty I_m(r) J_{2m}(rs) r\, dr \tag{32}$$

Making use of (29), each $H_m(s)$ can be expressed as a summation of the form,

$$H_m(s) = \sum_j h_{j,m}\tilde{\psi}_{2j,2m}(s). \tag{33}$$

Finally, using the orthogonality of the $\tilde{\psi}$ functions, we obtain the $h_{j,m}$ coefficients from

$$h_{j,m} = \frac{2^{2m+3}(m+j)!^2(2j)!U}{(2j+2m)!\Gamma(j+\frac{1}{2})^2} \int_0^\infty H_m(s)\tilde{\psi}_{2j,2m}(s) s\, ds \tag{34}$$

In general, these $h_{j,m}$ coefficients then allow us to determine the $p_{\ell,m}$ model power coefficients from the solution of a matrix linear equation at each value of $m = 0, 1, 2, \ldots,$

$$\mathbf{p} = \mathbf{C}^{-1}\mathbf{h} \tag{35}$$

or, in indicial notation, $h_j = C_{j,\ell}p_\ell$ where the constants in the coefficient matrix are given by the formula

$$C_{j,\ell,m} = \int_0^\infty \int_0^\infty \psi^2_{\ell,m}(r)\tilde{\psi}_{2j,2m}(s) J_{2m}(rs) rs\, dr\, ds, \tag{36}$$

which can be expressed as closed form expressions using (29, 34).

For the case of axisymmetric source electric fields, this procedure can be used to obtain all of the p_ℓ coefficients with $m = 0$, and $I_m = 0$ for all $m > 0$. To deal with more general sources, where non-orthogonal modes corresponding to different values of m are mixed together, some assumption must be made to relate modal powers. Some of the engineering literature suggests that effects not included in the linear model can cause interactions between modes belonging to the same degenerate mode group. It is believed that these effects will cause the power to equilibrate between all the modes in the same group [6, 8, 11, 12]. While it is not clear how this behavior can be justified, such a relation would be sufficient to extend the Hankel transform procedure to calculate all of the modal power coefficients $p_{\ell,m}$. Another approach to the general problem that would not involve extra assumptions on the distribution of power made use of a constrained least-squares fit to any available data points sampled from $I(r)$.

Finally, we close with some comments about the non-uniqueness of the electric field E that can be obtained from the power coefficients p_i. Recall from (9) that $p_i = |a_i|^2$. Thus a_i is determined only up a phase (if the coefficients are restricted to be real then $a_i = \pm\sqrt{p_i}$). Thus there exists an infinite family of $E_{in} = \sum_i \tilde{a}_i\psi_i$, where $\tilde{a}_i = \pm\sqrt{p_i}$ all producing the same intensity output $I(r)$. On one hand, this non-uniqueness may be viewed negatively as the inability to uniquely determine the input beam. On the other hand, that a variation of inputs yields the same output may present engineering advantages when designing high speed exciting lasers.

4 Conclusion

Other attempts at determining modal power from intensity measurements have been made. In [2] the underlying assumption of an incoherent source is used, so (13) holds. In [6], it is assumed that the phases of each mode are not correlated to one another. The results of Leminger and Grau [8] are consistent with ours. They also find that the modal power distribution cannot be uniquely recreated from the near-field intensity measurements. Piazzola and De Marchis [11], alternatively, do not mention the issue of non-uniqueness in their derivation of modal powers from near-field intensity. Other relevant works include [5, 12].

In this note, we have attempted to pinpoint potential difficulties in reconstructing the power distribution in a multi-mode fiber strictly from the near-field intensity read-out. The primary analytic work was performed on a quadratic index graded fiber, in the axi-symmetric case, for which closed form solutions in the form of Laguerre polynomials is available. The analysis indicates that non-uniqueness of this power distribution may be problematic. We have also identified situations in which the proposed $I(r)$ actually corresponds to $|E|^2$, none of which fall into the physical context being considered here. While our analysis was performed only on a simple fiber profiles and for relatively simple input beams, we do not expect that the concerns raised by our analysis will disappear for more complicated geometries.

References

[1] P. Beckmann. *Orthogonal polynomials for engineers and physicists.* Golem press, Boulder, CO, 1973.

[2] M. Calzavara, P. Di Vita and U. Rossi Mode power distribution measurements in optical fibres. *CSELF Report IX*, 5:447–451, 1981.

[3] P. K. Cheo. *Fiber Optics and Optoelectronics.* Prentice Hall, New Jersey, 1990.

[4] A. H. Cherin. *An introduction to optical fibers.* McGraw Hill, New York, 1983.

[5] P. Chu. *Calculation of impulse response of multimode optical fibre. Electronic Letters*, 16(11): 429-431, 1980.

[6] Y. Daido, E. Miyauchi, T. Iwama, and T. Otsuka. Determination of modal power distribution in graded-index optical waveguides from near-field patterns and its application to differential mode attenuation measurement. *Applied Optics*, 18(13):2207–2213, 1979.

[7] I. S. Gradshteyn and I. M. Ryzhik. *Table of Integrals, Series and Products.* Academic Press, Boston, 1994.

[8] O. G. Leminger and G. K. Grau. Near-field intensity and model power distribution in multimode graded-index fibres. *Electronics Letters*, 16(17):678–679, 1980.

[9] E.-G. Neumann. *Single-Mode Fibers.* Springer-Verlag, Berlin, 1988.

[10] T. Okoshi. *Optical Fibers.* Academic Press, New York, 1982.

[11] S. Piazzola and G. De Marchis. Analytical relations between model power distribution and near-field intensity in graded-index fibers. *Electronics Letters*, 15(22):721–722, 1979.

[12] J. Saijonmaa, A. B. Sharma, and S. J. Halme. Selective excitation of parabolic-index optical fibers by gaussian beams. *Applied Optics*, 19(14):2442–2452, 1980.

[13] A. W. Snyder and J. D. Love. *Optical Waveguide theory.* Chapman and Hall, London, 1983.