Bond Default Correlation

Problem posed by

Luis Reyna
Merrill Lynch

Report by

Ron Buckmire
Stephen Braun
Susan Cole
Matthew Schatz

Merrill Lynch sought to model the correlation of bonds defaulting in a given time period. Two models were discussed by Dr. Reyna. The first involved the modeling of the value of a company as a random stochastic process with (in the simplest case)

$$\frac{\Delta V}{V} = r\Delta t + \sigma\epsilon(\Delta t)^{1/2}$$

where

V=value of company
$r\Delta t$=mean of distribution for $\Delta V/V$
ϵ= random draw from a Gaussian distribution with mean equal to 0 and standard deviation equal to 1
$\sigma(\Delta t)^{1/2}$=standard deviation of distribution for $\Delta V/V$.

The bond was said to default when the value of the company equaled the cost of the outstanding bond debt. This model predicts exponentially small default rates for small time and was deemed unacceptable by Dr. Reyna as the observed bond default rate curve is observed to have a nonzero slope for small time. See Fig. 1.

The second model presented by Dr. Reyna used a Poisson process approach to model the probability of default of an individual bond. In this model

$$\Delta\lambda = \alpha\Delta t + \sigma\epsilon(\Delta t)^{1/2}$$

where

λ=instantaneous rate of default for a given bond
$\alpha\Delta t$=mean of distribution for $\Delta\lambda$
ϵ=random draw from a Gaussian distribution with mean equal to 0 and standard deviation equal to 1
$\sigma(\Delta t)^{1/2}$ = standard deviation of distribution for $\Delta\lambda$.

The probability of survivial in a Poisson process is given by

$$P = e^{-\int_0^t \lambda(s)\,ds}$$

and the probability of default is defined to be

$$D = 1 - e^{-\int_0^t \lambda(s)\,ds}.$$

For the simple case where α and σ are constant,

$$D = 1 - e^{-\alpha t^2/2 - \lambda_0 t + \sigma^2 t^3/6}.$$

This implies that as t increases the probability of default becomes negative. Small random contributions to the instantaneous rate of default yield exponentially small contributions to D when they are positive and exponentially large contributions to D when they are negative. This model also does not yield the desired behavior of Fig. 1.

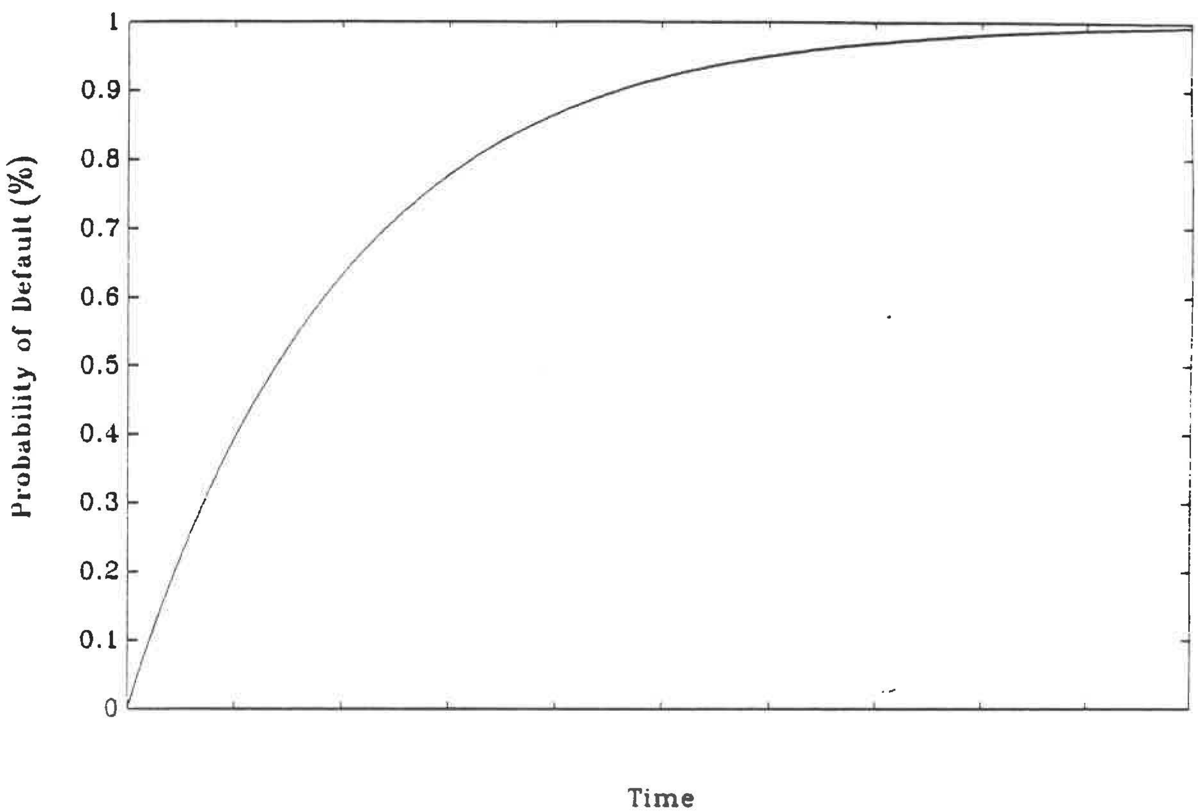

Figure 1: Typical expected bond rate default curve.

We chose, therefore, to formulate a deterministic Poisson process of individual bond default behavior that mimics the behavior of Fig. 1 and allow randomness to enter in the correlation of multiple defaults. Although the number of bonds changes from year to year the total number of bonds defaulting is typically less than 50 per year. This means that there is very little bond default data available on which to perform extensive correlations and therefore any factor model must be quite simple.

In order to identify possible factor(s) we used the historical default rates given in Moody's [1]. The one-year default rates from 1970 to 1995 are summarized in Table 1.

Table 1. One-year Default Rates by Bond Class from 1970-1995

Aaa	.00%
Aa	.03%
A	.01%
Baa	.13%
Ba	1.42%
B	7.62%

The probability of default is highly dependent on the Moody's bond rating. Since the default rate of investment grade bonds (Baa and above) is so low, only Ba and B rated bonds are modeled below. Fig. 2 shows the default rate of Ba and B bonds from 1971 to 1993. (Data from Moody's [1].)

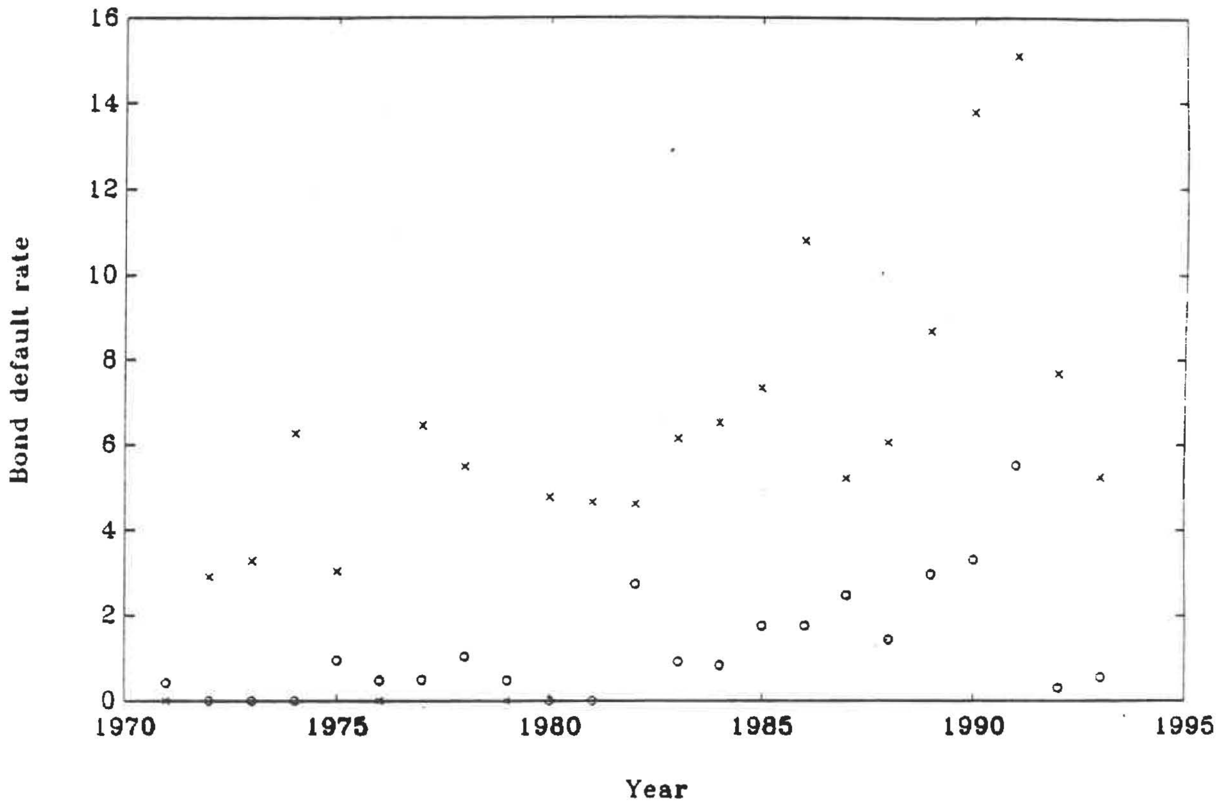

Figure 2: Default rate of Ba (o) and B (x) bonds from 1971 to 1993.

We modeled the default rate of an individual bond as dependent on the Moody's bond rating, a credit climate factor γ and an industrial factor $I(t)$. The probability of default of an individual bond was chosen to satisfy

$$D = 1 - e^{-\int_0^t \lambda(s)\,ds} = 1 - e^{-(a+b\gamma)t - I(t)}$$

where a and b are constants which depend on the bond rating. In order to identify the credit climate factor γ we set $I(t) = 0$ and chose

$$\gamma = \alpha T - C$$

where T is the Treasury bill rate, C is the annual percentage change in the Consumer Price Index as taken from Sharpe [2] and α is a paramter chosen to minimize the least squares error as discussed below. In a Poisson process, events (defaults) are assumed to occur independently. The expected value of the rate of failure of N bonds with the same probability P is NP and the average default rate is $NP/N = P$. Thus, the historical default rates define the probability of default of a single member of the class. We chose a, b and γ so as to minimize the mean square error

$$E = \sum_{i=1}^{23} \left[-\ln(1 - D_i) - (a + b(\alpha T_i - C_i)) \right]^2$$

where the data is taken over the 23 years from 1971 through 1993. For α fixed, this problem is equivalent to a standard linear regression analysis. Fig. 3 shows the least squares error as a function of α for Ba and B rated bonds. The optimal a, b and α for Ba bonds was found to be $a = .0168$, $b = .00215$ and $\alpha = .5343$. The optimal for B bonds was found to be $a = .0708$, $b = .00514$ and $\alpha = .5526$. Fig. 4 shows the optimal least squares fit for Ba and B rated bonds.

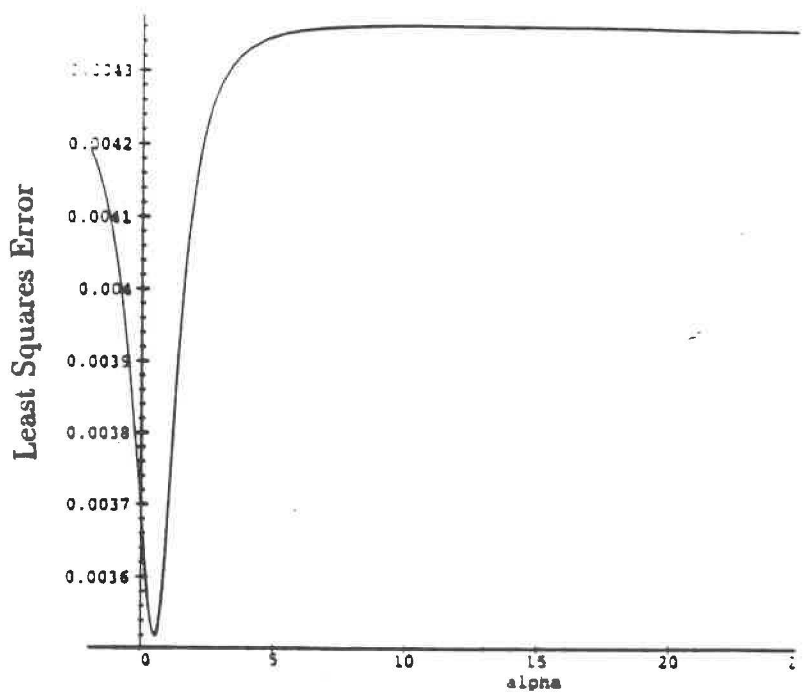

Figure 3a: Least squares error as a function of α for Ba bonds.

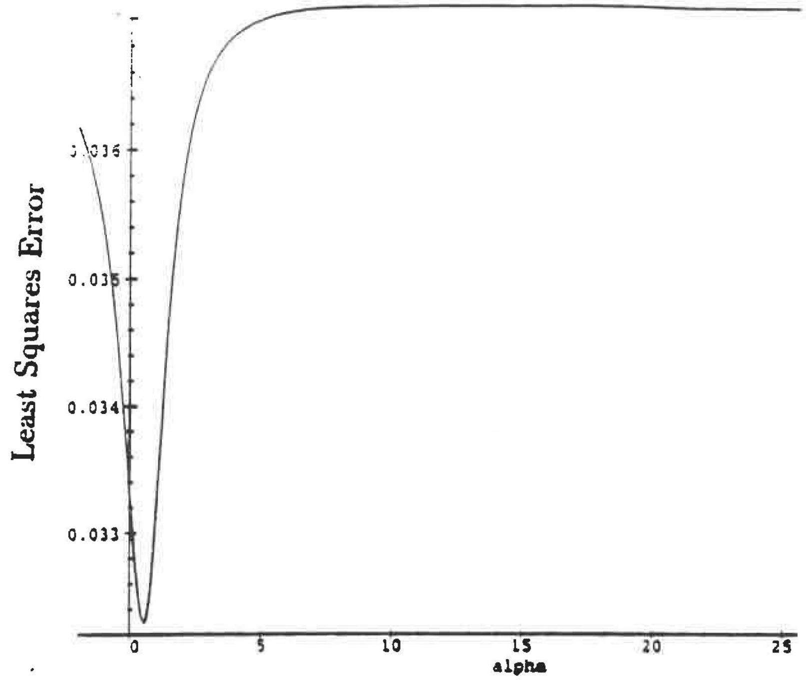

Figure 3b: Least squares error as a function of α for B bonds.

According to this model the probability of default of any particular bond in a given class is deterministic. However, the probability of N failures out of M bonds is given by the Binomial distribution

$$B(N; M, D) = \binom{M}{N} D^N (1-D)^{M-N}.$$

Fig. 5a shows a plot of $B(N; M, D)$ with M=250 for Ba rated bonds with $\gamma = -5, 0, 5$. Fig. 5b shows the corresponding plot for B bonds. Note that as γ increases, these distributions spread out. For each value of γ, a confidence interval can be defined that encloses a central percentage of the Binomial distribution. For example, given $M = 250$, a confidence level of 90% and a fixed value of γ (say $\gamma = -5$), begin adding up the probabilities of default corresponding to $N = 0, 1, \ldots$defaults until they first total at least 5%. Fix N_l to be the stopping value of N. Then $B(N_l; M, D)$ is the lower bound corresponding to the N_l in this confidence level. To find the upper bound, add up the probablities of default corresponding to $N = M, M-1, \ldots$ until they first total at least 5%. Fix N_u to be the stopping value of N. Then $B(N_u; M, D)$ is the upper bound corresponding to N_u in this confidence level. The solid (mean) line in Fig. 6 represents the least squares fit for defaults on bonds as a function of the factor γ based on Moody's [1] default observations (shown as small circles). For any fixed γ, the innermost steps surrounding the line corresponds to the set of N that lie within the middle 90% of the default distribution. The next set of steps

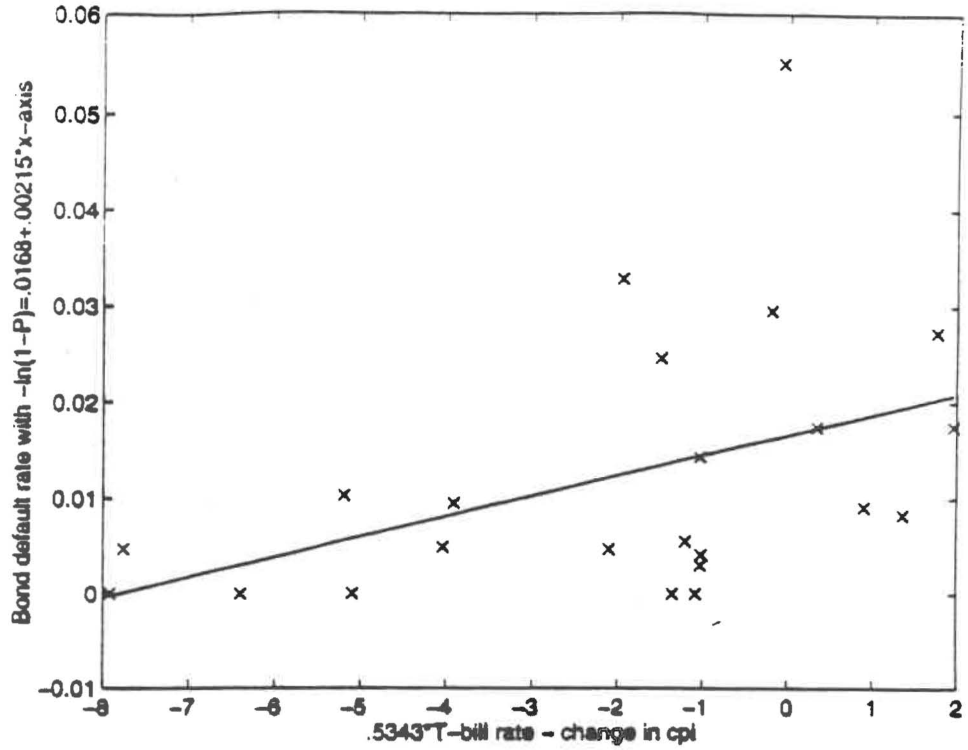

Figure 4a: Least squares fit for Ba bonds.

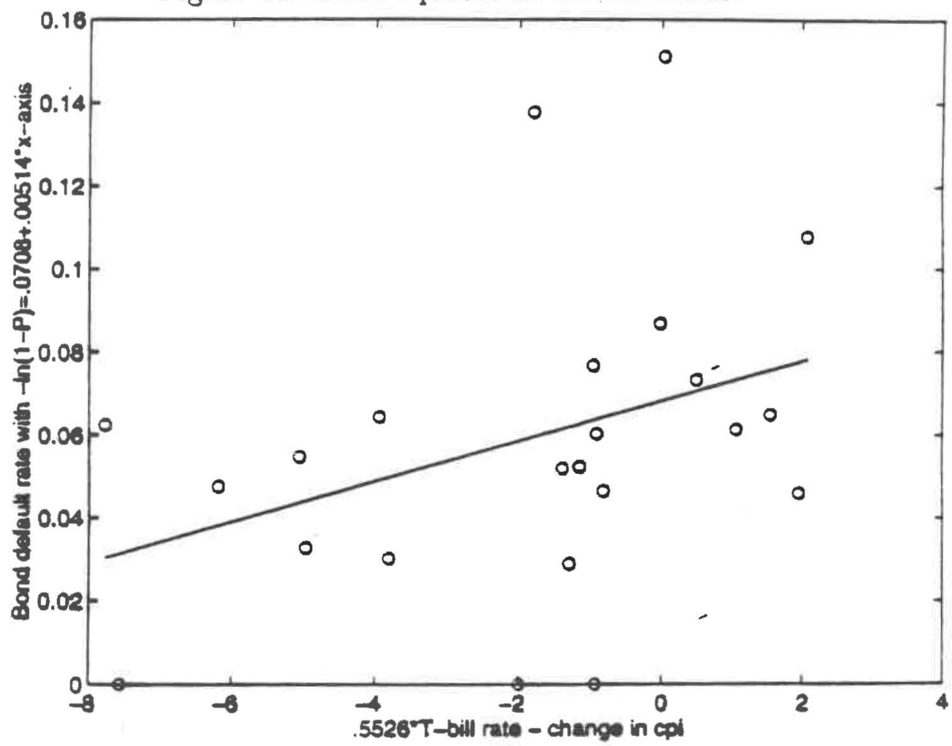

Figure 4b: Least squares fit for B bonds.

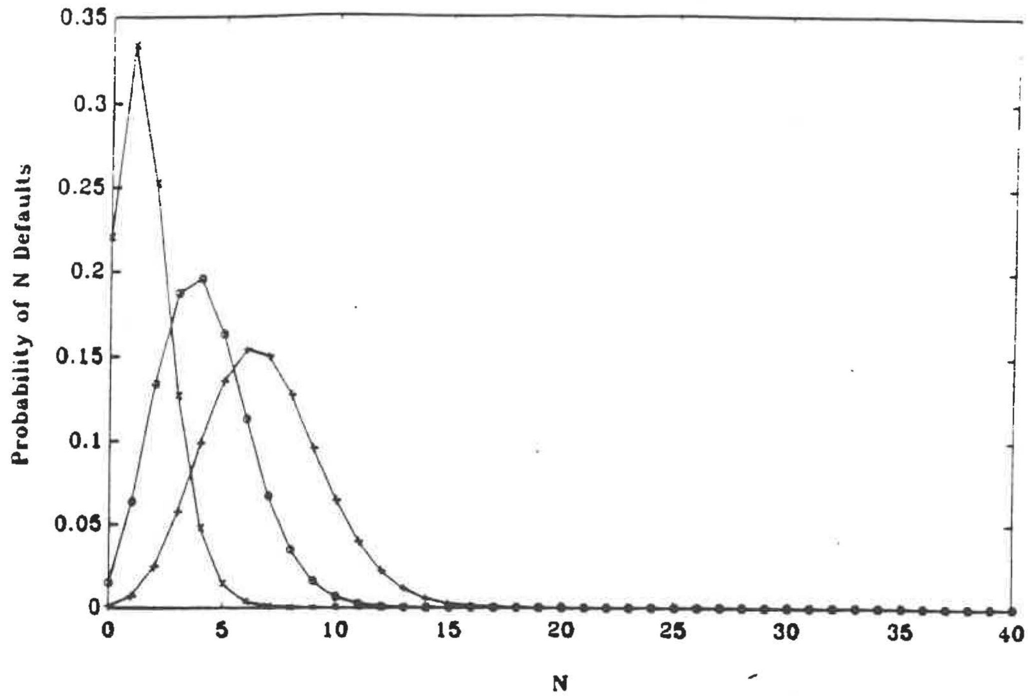

Figure 5a: $B(N; M, D)$ with M=250 for Ba rated bonds with γ=-5 (x), 0 (o) and 5 (+).

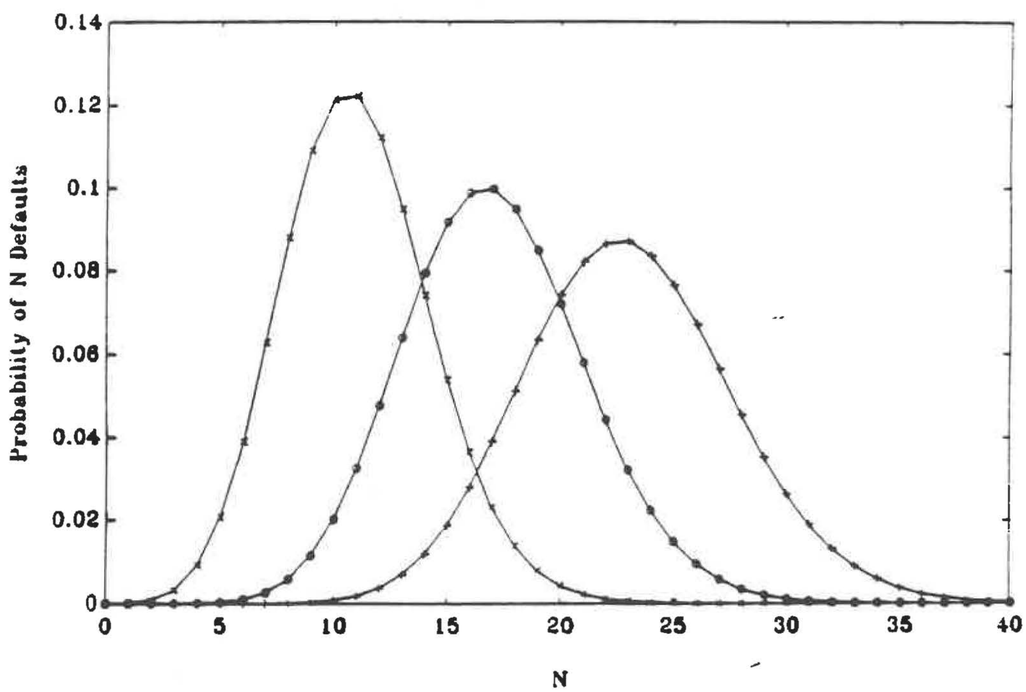

Figure 5b: $B(N; M, D)$ with M=250 for B rated bonds with γ=-5 (x), 0 (o) and 5 (+).

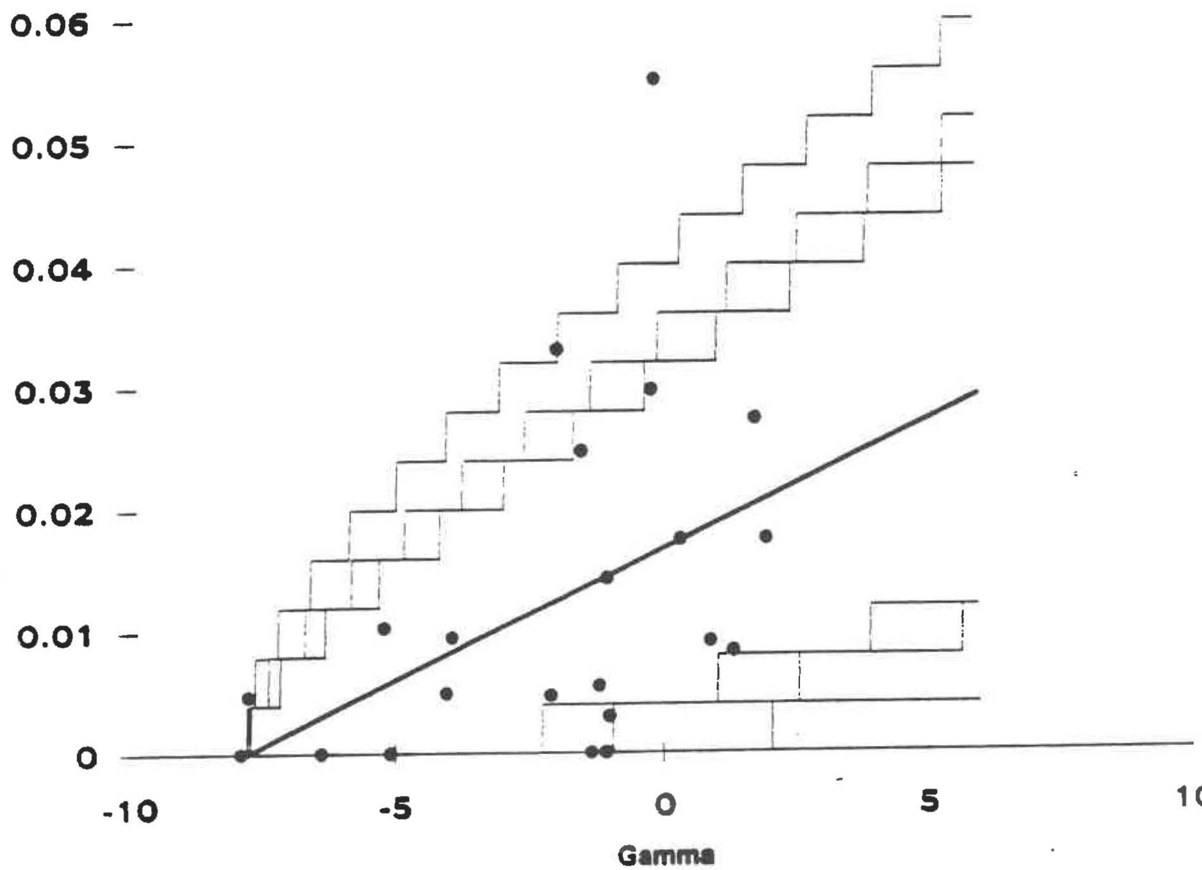

Figure 6a: Observed bond defaults and confidence intervals for Ba rated bonds with M=250.
Straight Line: Least Squares Fit. Small Circles: 1971 - 1993 Observations.
Innermost Steps: 90% Confidence Interval
Middle Steps: 95% Confidence Interval
Outermost Steps: 99% Confidence Interval

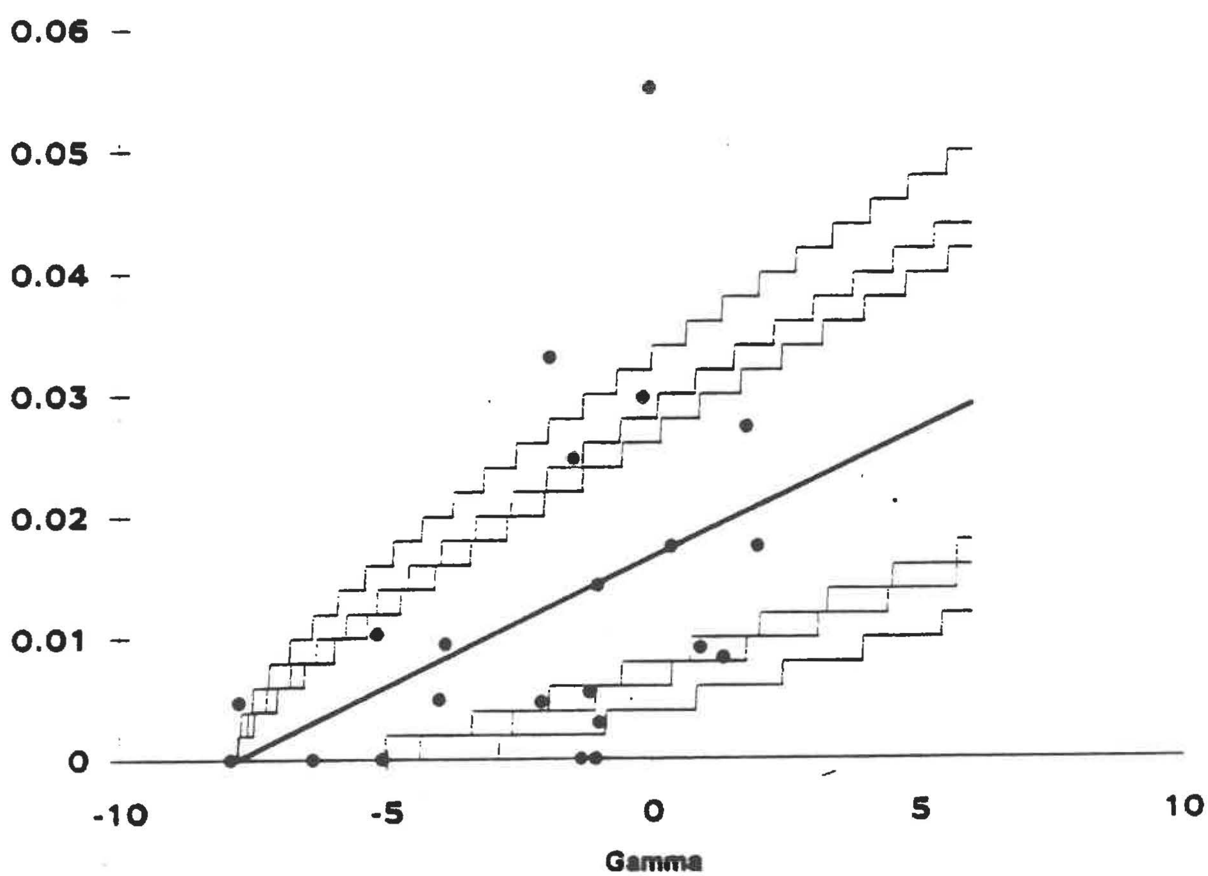

Figure 6b: Observed bond defaults and confidence intervals for Ba rated bonds with M=500.
Straight Line: Least Squares Fit. Small Circles: 1971 - 1993 Observations.
Innermost Steps: 90% Confidence Interval
Middle Steps: 95% Confidence Interval
Outermost Steps: 99% Confidence Interval

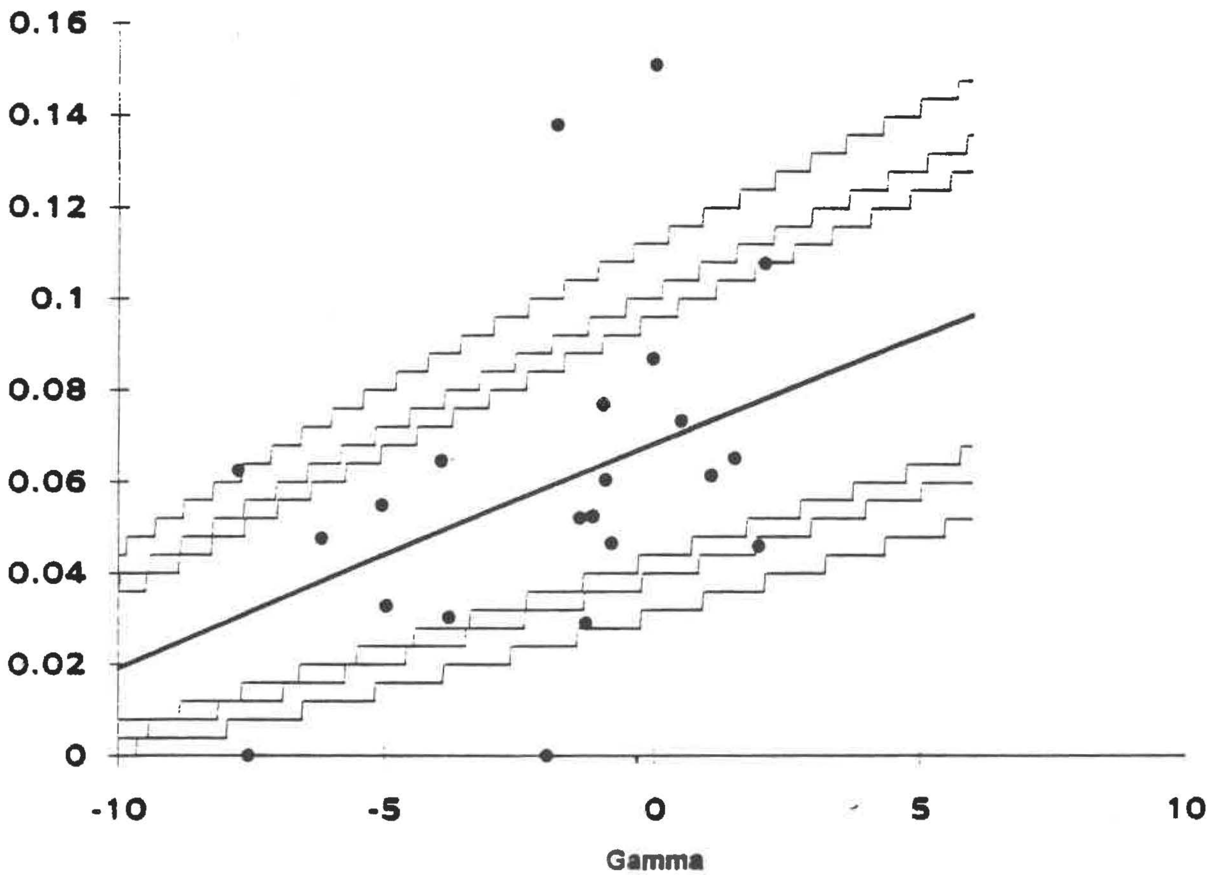

Figure 6c: Observed bond defaults and confidence intervals for B rated bonds with M=250.
Straight Line: Least Squares Fit. Small Circles: 1971 - 1993 Observations.
Innermost Steps: 90% Confidence Interval
Middle Steps: 95% Confidence Interval
Outermost Steps: 99% Confidence Interval

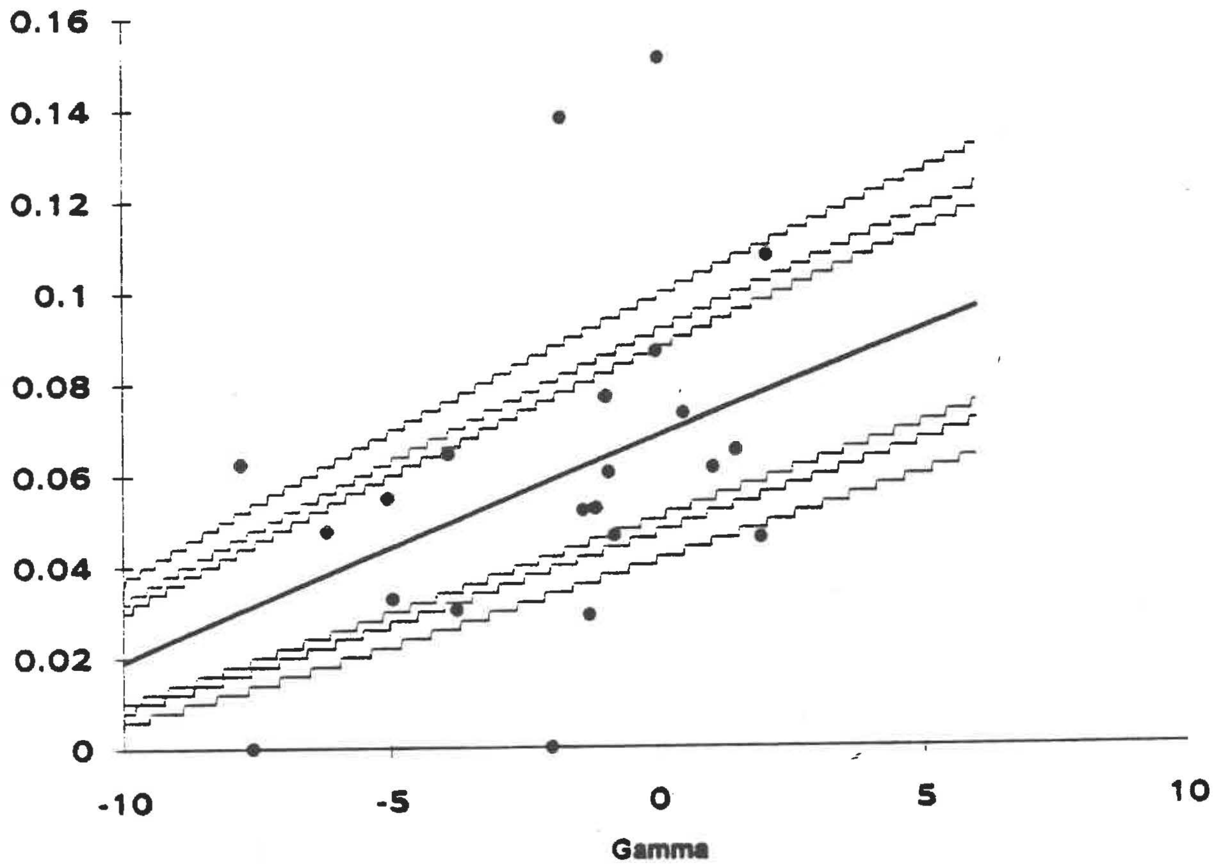

Figure 6d: Observed bond defaults and confidence intervals for B rated bonds with M=500.
Straight Line: Least Squares Fit. Small Circles: 1971 - 1993 Observations.
Innermost Steps: 90% Confidence Interval
Middle Steps: 95% Confidence Interval
Outermost Steps: 99% Confidence Interval

corresponds to the set of N that lie within the middle 95% of the default distribution. The outermost set of steps corresponds to the set of N which lie within the middle 99% of the default distribution. Note these confidence intervals widen as γ increases (within the range plotted) since the default distributions in Fig. 5 spread out as γ increases. The number of Ba and B rated bonds are typically a few hundred each. Note that the curves in Fig. 6 fit the observed trend reasonably well with the exception of a few outlying cases. The cases where the defaults are too low are not worrisome. Nobody minds being paid. However, the cases with higher than predicted defaults must be investigated carefully. The model results in Fig. 6 include only the effects of the credit climate paramter γ. It is expected that knowledgeable analysists could also incorporate information into the industrial factor $I(t)$. During "normal" times $I(t)$ would be zero for all industries. During times which are difficult for a particular industry, (for example, the clothing industry in recent years due to the aging of baby boomers who no longer buy so many clothes), a nonzero value for $I(t)$ can be assigned based on the default rates under similar past circumstances. In this way, abnormally high expected default rates can be modeled.

References

[1] Carty, L. V., and Lieberman, D., *Corporate Bond Defaults and Default Rates 1938-1995*, Moody's Investors Service Global Credit Research, 1996.

[2] Sharpe, W. F., Alexander, G. J. and Bailey, J. V., *Investments*, Prentice Hall, 1995.