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The optimization of a flexible cable

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1 Introduction

The mechanics of computer hard disk drives is an important industrial problem which favors simple designs that can be optimized within a mathematical framework. Hard drives must perform in a very efficient manner; with maximum speed in accessing data, high accuracy in operation and low power consumption. Additionally, all components must be designed to maintain these conditions over thousands or millions of repetitions for a product lifespan of several years. These considerations have produced current designs with a minimum of moving mechanical parts, and each of those parts are themselves relatively simple and reliable. A support arm positions the read-write head over the rotating disk to access data. To access all of the tracks on the disk, the arm can be rotated quickly through a 30° range of angles by a very accurate actuator motor. Connected to this rotating structure is a flexible computer ribbon cable that carries data to and from fixed electronic components in the rest of the computer. If this cable were very stiff it would impair the motion of the actuator slowing track-to-track data access speeds and might produce errors in track positioning. If this cable were very loose then the resulting long cable would not easily fit into small disk drive design geometries. We now present a mathematical formulation for determining the shape of the flexible cable and describe the design optimization problem.

[1]Proposed by Roger Wood, Storage Systems Division, IBM Research, San Jose CA

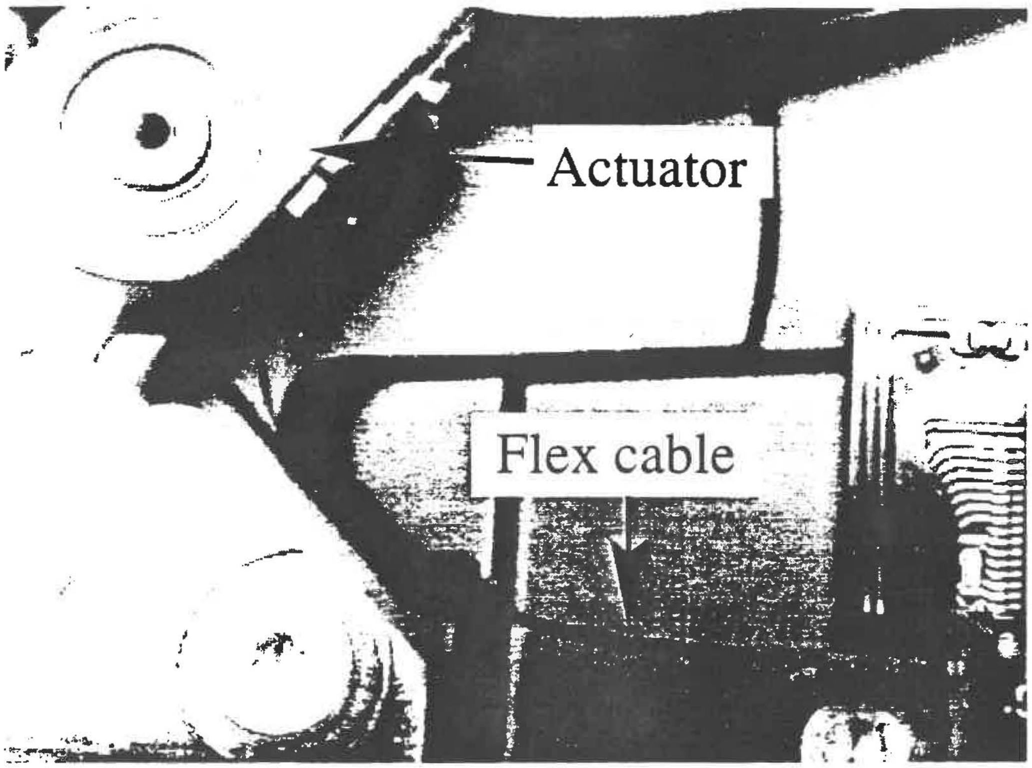

Figure 1: A photograph of a flex cable in a hard disk drive.

2 Formulation

The equation for the equilibrium shape of the inextensible flex cable is derived from the general equation for the bending of rods by end-point forces given in Landau and Lifshitz [4],

$$EI\frac{d\mathbf{R}}{dS} \times \frac{d^3\mathbf{R}}{dS^3} = \mathbf{F} \times \frac{d\mathbf{R}}{dS}, \tag{1}$$

where $\mathbf{R}(S)$ is the shape of the bent rod in terms of the arclength, and $E \cdot I$ is the bending stiffness, Young's modulus times the moment of inertia. $\mathbf{F}$ is the vector for the pair of equal and opposite forces applied at the endpoints of the rod or cable. The flex cable is constrained to bend in one plane, consequently we can write the tangent to the cable and the force as

$$\frac{d\mathbf{R}}{dS} = (\cos\theta, \sin\theta, 0)^{\mathsf{T}}, \qquad \mathbf{F} = (-F\cos\psi, -F\sin\psi, 0)^{\mathsf{T}}, \tag{2}$$

where $\theta = \theta(s)$ is the angle of inclination of the cable and ψ is the angle of the force. Thus, equation (1) reduces to a nonlinear pendulum equation for the angle of inclination of the cable [5],

$$EI\frac{d^2\theta}{dS^2} + F\sin(\theta - \psi) = 0, \tag{3}$$

along with the parametric equations for the shape of the cable

$$\frac{dX}{dS} = \cos\theta, \qquad \frac{dY}{dS} = \sin\theta. \tag{4}$$

To complete the description of the cable, we must specify the positions and orientations of the endpoints. One end is clamped at a fixed position, taken to be the origin, without loss of generality,

$$\theta(0) = 0, \qquad X(0) = 0, \qquad Y(0) = 0. \tag{5}$$

The other end of the cable is clamped to the actuator, modeled as a rotating cylinder with center $(\bar{X}, \bar{Y})$, radius R, and the rotation angle α,

$$\theta(L) = \alpha - \pi/2, \qquad X(L) = \bar{X} + R\cos\alpha, \qquad Y(L) = \bar{Y} + R\sin\alpha, \tag{6}$$

where the boundary condition on θ specifies tangential attachment to the cylinder. Equations (3–6) describe a one-parameter family of nonlinear two-point eigenvalue problems; for a continuous range of rotation angles α there is a family of solutions with force (F, ψ) that satisfy the geometric constraints given by $(\bar{X}, \bar{Y}, R, L)$.

We non-dimensionalize the problem by the length of the cable,

$$X = Lx, \qquad Y = Ly, \qquad R = Lr, \qquad S = Ls, \tag{7}$$

and get a nondimensional parameter for the ratio of the applied force to the bending stiffness,

$$\lambda = \frac{FL^2}{EI}. \tag{8}$$

This yields the boundary value problem,

$$\frac{d^2\theta}{ds^2} + \lambda \sin(\theta - \psi) = 0, \qquad 0 \leq s \leq 1 \tag{9}$$

$$\frac{dx}{ds} = \cos\theta \qquad \frac{dy}{ds} = \sin\theta \tag{10}$$

$$\begin{array}{ll} \theta(0) = 0 & \theta(1) = \alpha - \pi/2 \equiv \theta_\ell \\ x(0) = 0 & x(1) = \bar{x} + r\cos\alpha \equiv x_\ell \\ y(0) = 0 & y(1) = \bar{y} + r\sin\alpha \equiv y_\ell \end{array} \tag{11}$$

Given the length of the cable, the radius and position of the actuator $(r, \bar{x}, \bar{y})$, and the desired attachment angle α, we can solve this system to yield the shape of the cable $(\theta(s), x(s), y(s))$ and the necessary applied force, the eigenvalues (λ, ψ).

The form of problem (9–11) can be clarified by re-casting it as an initial value shooting problem. Using the trigonometric identity for $\sin(\theta - \psi)$ in (9) yields

$$\frac{d^2\theta}{ds^2} + \lambda\left(\cos\psi\frac{dy}{ds} - \sin\psi\frac{dx}{ds}\right) = 0, \tag{12}$$

with the use of equation (10). Integrating (12) from s to $s = 1$ yields a first order equation that explicitly incorporates the boundary conditions at $s = 1$,

$$\frac{d\theta}{ds} = \lambda\left([x - x_\ell]\sin\psi - [y - y_\ell]\cos\psi\right) + \theta'_\ell, \tag{13}$$

$$\frac{dx}{ds} = \cos\theta, \tag{14}$$

$$\frac{dy}{ds} = \sin\theta, \tag{15}$$

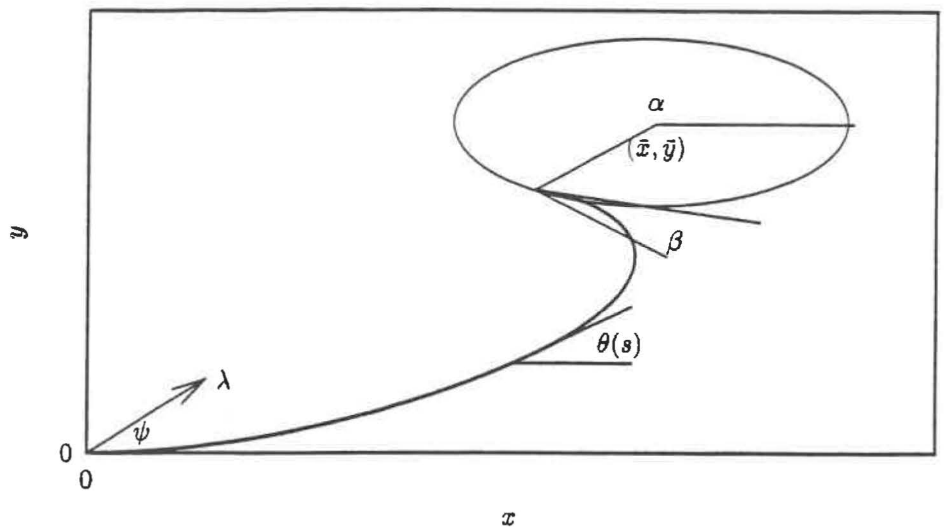

Figure 2: The geometry of the basic problem.

where $\theta'_\ell \equiv \theta'(1)$,

$$x(0) = 0, \qquad y(0) = 0, \qquad \theta(0) = 0. \tag{16}$$

Equations (13–15) form a third order nonlinear system with zero initial values (16). If we denote a solution of this system as $\mathbf{u}(s) = (\theta(s), x(s), y(s))^\mathsf{T}$, then we are seeking a solution of

$$\mathbf{G}(\mathbf{u}, \mathbf{\Lambda}; \alpha) \equiv \begin{pmatrix} \theta(1) - \theta_\ell \\ x(1) - x_\ell \\ y(1) - y_\ell \end{pmatrix} = \mathbf{0}, \tag{17}$$

where the endpoint value of the solution $\mathbf{u}(1)$ depends on the vector of parameters, $\mathbf{\Lambda} = (\lambda, \psi, \theta'_\ell)^\mathsf{T}$. This form of the problem can be solved numerically using Newton's method. Starting from a reasonable initial guess for the parameter values, $\mathbf{\Lambda} = \mathbf{\Lambda}_0$, define the residual

$$\mathbf{G}_n = \mathbf{G}(\mathbf{u}_n, \mathbf{\Lambda}_n; \alpha), \qquad n = 0, 1, 2, \ldots \tag{18}$$

and the discrete (finite-difference) approximation of the Frechet derivative or Jacobian as

$$\mathbf{J}_n = \left.\frac{\delta \mathbf{G}}{\delta \mathbf{\Lambda}}\right|_{(\mathbf{u}_n, \mathbf{\Lambda}_n)}, \tag{19}$$

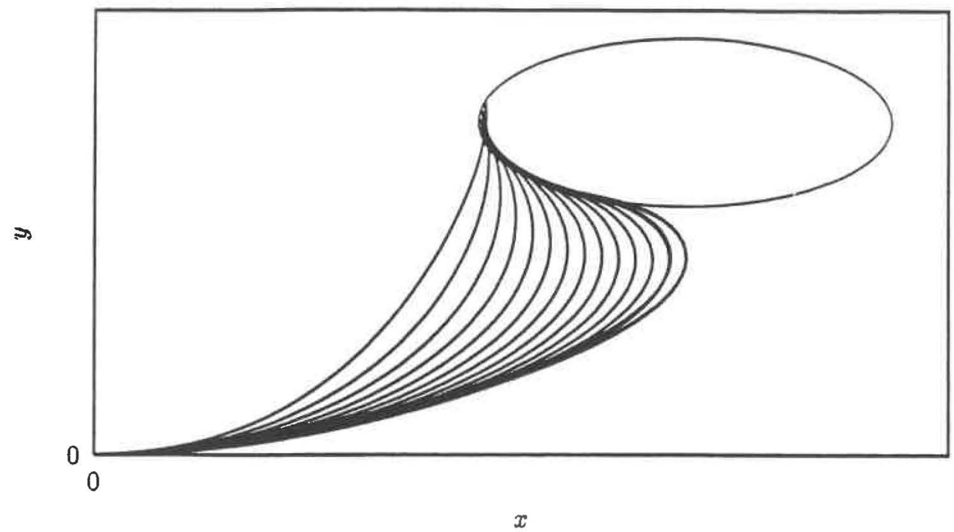

Figure 3: The shape of the flexible cable as the actuator is rotated through a range of angles.

then iterates of the parameters are given by

$$\mathbf{J}_n(\mathbf{\Lambda}_{n+1} - \mathbf{\Lambda}_n) = -\mathbf{G}_n. \tag{20}$$

The sequence $\{\mathbf{\Lambda}_n\}$ will converge to a solution quadratically if the initial guess $\mathbf{\Lambda}_0$ is sufficiently close to the final solution.

Of primary interest in this problem is the effect of the cable on the functioning of the actuator. It is desired that the torque or moment on the actuator due to the cable be minimized over a typical angular range of $\pi/6$ for α. This applied torque is

$$M = -\left(\theta'_\ell + \lambda r \sin(\alpha - \psi)\right). \tag{21}$$

Once a solution of (13–16) has been obtained, we can calculate its applied torque.

3 Phase plane formulation and elliptic functions

Briefly returning to equation (9), we note that the change of variables

$$\tilde{s} = s\sqrt{\lambda}, \qquad \phi(\tilde{s}) = \theta(s) - \psi, \tag{22}$$

yields the pendulum equation for $\phi(\bar{s})$,

$$\frac{d^2\phi}{d\bar{s}^2} + \sin\phi = 0. \tag{23}$$

The first integral of this equation is the energy

$$E = \frac{1}{2}\phi'^2 - \cos\phi, \tag{24}$$

which yields the integral curves for the (ϕ, ϕ') phase plane. The solutions of (9, 10, 11) with fixed design geometry $(\bar{x}, \bar{y}, r)$ at a given value of α will be finite segments of these curves. The endpoints of the segment are determined by the details of the boundary value problem and generally do not have a clear interpretation in the phase plane. Nor does the separatrix, the solution that separates two different classes of solutions in the phase plane, seem to have a clear significance for the boundary value problem. As the actuator angle α is varied over a finite range, the solution changes integral curves in a continuous manner, sweeping out a region in the phase plane. It should be noted that this region will have define bounds corresponding to physical limitations on the minimum and maximum realizable values of α. This α-family of solutions is not necessarily locally invertible, as the region covered in the phase plane can fold back onto itself. Physically this means that given a sub-segment of a solution in the phase plane may not uniquely define the entire cable configuration since that portion of the integral curve might be contained in several distinct solutions that have different endpoints.

One class of solutions that have a simple geometric description is the set of inflexional and non-inflexional solutions described by Love [5]. These are symmetric segments in the phase plane corresponding to cables with endpoint forces that act on a single line of force. While it is true that all solutions of the pendulum equation (23) can be expressed analytically in terms of elliptic functions [1, 3, 2], the resulting expressions can be somewhat cumbersome to deal with. However the inflexional solutions yield relatively simple solutions that prove to be useful.

The term "inflexional" refers to the fact that the endpoints of the cable for these solutions are inflection points where $\theta'(s) = 0$ and the curvature changes sign. There is a continuous one-parameter family of these solutions, described by the angle ψ between the cable and the line of force. In terms of ψ, the modulus for the Jacobian elliptic functions is $k = \sin(\frac{1}{2}\psi)$. As described by Love [5], in terms of the magnitude λ and angle ψ of the applied force, the length of the cable for an inflectional solution is

$$\ell = \frac{2}{\sqrt{\lambda}} K(k) = 1, \tag{25}$$

where $K(k) = F(\pi/2|k)$ is the complete elliptic integral of the first type. Similarly, the end-to-end distance of the cable is

$$d - r = \frac{2}{\sqrt{\lambda}}(2E(k) - K(k)), \qquad d = \sqrt{\bar{x}^2 + \bar{y}^2}, \tag{26}$$

where $E(k) = E(\pi/2|k)$ is the complete elliptic integral of the second type. Observe that we can determine the angle ψ given the ratio of these two lengths,

$$0 < d - r = \frac{2E(k)}{K(k)} - 1 < 1, \tag{27}$$

and consequently we can obtain the magnitude of the force,

$$\lambda = \left(\frac{2nK(k)}{\ell}\right)^2, \tag{28}$$

where $n = 1$ is the basic solution of the problem and other "higher order harmonic" solutions exist for all $n = 2, 3, 4, \ldots.$ From a geometric construction it can be shown that for these symmetric solutions the actuator angle is given by $\alpha = 2\psi$. Similarly from geometry, we can determine the actuator position,

$$\bar{x} = (d - r)\cos\psi, \qquad \bar{y} = (d - r)\sin\psi. \tag{29}$$

Therefore, if we are given the force (λ, ψ) and the fact that the cable is an inflectional shape, then we can uniquely determine all of the remaining parameters.

In general for a given actuator and cable, these solutions cannot be achieved; they occur only for a very restricted (one-parameter) set of problems. However, they are very helpful as initial conditions for using Newton's method to solve system (13–16) to converge to solutions defined by sets of parameters that are close to the inflectional values.

Another consideration that we mention now, is that for real hard drives, the attachment boundary condition at $s = 1$ given in (11) is somewhat simplified. It assumed that the cable is clamped to the actuator motor tangent to a circle. In reality, the cable is clamped to the arm of the actuator, a structure connected to the motor. The point of attachment still moves on

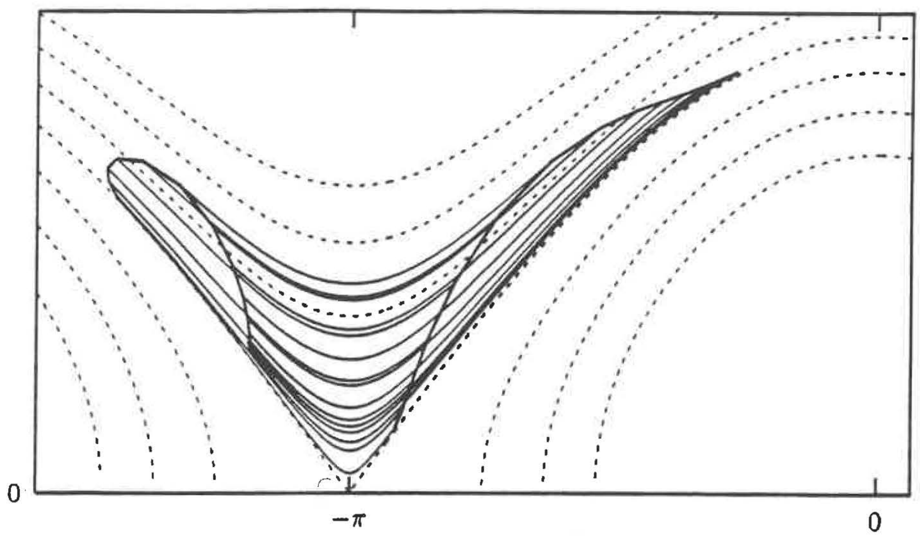

Figure 4: The cable at various angles α in the phase plane.

a circle as the motor turns, but its angle of attachment is generally an extra degree of freedom. Consequently, we introduce the defect angle β, in the boundary condition,

$$\theta(1) = \alpha + \beta - \pi/2. \tag{30}$$

4 Design optimization

In this section we begin the discussion of a rational design process to select a "best" cable and actuator configuration. As was described above, the primarily criterion in defining a best design is reducing the torque applied by the cable on the actuator,

$$M(\alpha) = -(\theta'_\ell + \lambda r \sin(\alpha - \psi)). \tag{31}$$

Any discussion of an optimization problem must be prefaced by a very clear statement of how the optimization process is to be carried out, i.e. which design parameters are prescribed and which parameters can be varied to seek the best design. In operation, to access all tracks on the hard disk, the actuator must turn through a range of 30 degrees. As a first problem, our goal is to find the optimal 30° range of angles for the design shown in figure 1, with the parameters

$$\bar{x} = 67 \text{ mm}, \ \bar{y} = 49 \text{ mm}, \ \ell = 84 \text{ mm}, \ \beta = 17^\circ, \ r = 17\text{mm}$$

are given and fixed. This optimization problem has a clear interpretation – the actuator, cable length and fixed endpoint are pre-defined, solutions for the flex cable satisfying these constraints exist for a range of rotation angles α – what is the 30° sub-interval in α that minimizes some measure of the moment, like the average or some other norm of $M(\alpha)$, on the 30° range? The solution of this problem will define an optimal operating angle α_0 which determines the best orientation of the actuator arm and fixed endpoint relative to the position of the hard disk[2]. This is the most restricted optimization problem; more general problems can be addressed by relaxing constraints on the position of the fixed endpoint.

References

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[2] One is reminded of jokes involving holding a lightbulb and turning the earth.