

Annular Two-Phase Flow

Problem Presented by

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1. Introduction

The problem presented to the workshop by David Edwards and Darton Strayer of the Knolls Atomic Power Laboratory (K.A.P.L.) concerned annular two-phase flow. It is observed experimentally that when large volumes of high speed vapor and smaller volumes of slower moving liquid flow in many situations (e.g., between horizontal or vertical parallel plates, or in horizontal or vertical tubes), the liquid forms a thin layer along the walls with the vapor flowing in the core. We wish to understand the dynamics of this flow, in particular, why this density stratification, which should be unstable, persists under a wide variety of flow conditions. This is summarized succinctly by the question, "Why doesn't the liquid rain?"

At present a code is used to solve averaged equations for the void fraction α , the averaged velocities $\mathbf{v}_g$ and $\mathbf{v}_\ell$ and the pressures p_g and p_ℓ of the gas and liquid respectively. The equations take the usual form

$$[(1 - \alpha)\rho_\ell]_t + \nabla \cdot (1 - \alpha)\rho_\ell \mathbf{v}_\ell = 0$$

$$[\alpha\rho_g]_t + \nabla \cdot \alpha\rho_g \mathbf{v}_g = 0$$

$$[(1 - \alpha)\rho_\ell \mathbf{v}_\ell]_t + \nabla \cdot (1 - \alpha)\rho_\ell \mathbf{v}_\ell \mathbf{v}_\ell = \nabla \cdot (1 - \alpha)(-p_\ell \mathbf{I} + \boldsymbol{\tau}_\ell + \boldsymbol{\tau}_\ell^{Re}) + (1 - \alpha)\rho_\ell \mathbf{g} + \mathbf{M}_{\ell g}$$

$$[\alpha\rho_g \mathbf{v}_g]_t + \nabla \cdot \alpha\rho_g \mathbf{v}_g \mathbf{v}_g = \nabla \cdot \alpha(-p_g \mathbf{I} + \boldsymbol{\tau}_g + \boldsymbol{\tau}_g^{Re}) + \alpha\rho_g \mathbf{g} + \mathbf{M}_{g\ell}$$

where the term M_{lg} ($= -M_{gl}$) is important, as it may be thought of as representing all the interface interaction terms that have been formally left out as a result of the averaging process. To close the model so that numerical solutions may be carried out, constitutive assumptions must be made for the term M_{lg} , τ_k , and τ_k^{Re} . The specific modelling that is involved must faithfully reflect the nature of the flow, and in particular must be appropriate to the two-phase flow regime that pertains.

For some two-phase flow regimes, suitable forms of M_{lg} are fairly well-known. For example, for bubbly flow where the total bubble fraction is not too high satisfactory models are available. (For this and other examples see Drew & Wallis (1996).) For annular flow, however, the position is much less clear. One way to do this is to include a somewhat artificial 'lift force' whose function is to allow a liquid film to exist stably on the top surface of a horizontal duct. Unfortunately, it is difficult to justify the large value of the lift coefficient that must be used to maintain the film, and there are cases where the code predicts that the film will not remain attached to the wall if only small changes in geometry are made. Evidently, a better understanding of the flow is required. The main questions that K.A.P.L. wanted to address in the MPI workshop were therefore

- (1) What are the mechanisms that maintain the annular film?
- (2) What are the mathematical descriptions of such mechanisms?
- (3) Can such mechanisms be used to demonstrate the stability of annular flows?

2. Annular Two-Phase Flow

Space does not permit anything more than a cursory discussion of the annular two-phase flow regime, but it may be thought of as being the predominant flow pattern for evaporators, many different types of boiler, natural gas pipelines and general steam heating systems. The annular flow regime is characterised by a fast-flowing central gas core surrounded by a liquid film attached to the walls of the tube or duct in which flow takes place. The gas core may or may not contain droplets and mass exchange in the form of droplet entrainment and redeposition is frequently present. Throughout the following discussion, unless otherwise indicated, two-dimensional cartesian duct flow (either vertical or horizontal) is examined (so that the added complications of cylindrical flow may be ignored) and, for simplicity, the flow is assumed to be adiabatic. Naturally, in most boilers and steam systems the heat transfer is the very point at issue; nevertheless annular two-phase flows are readily observed

in adiabatic systems. It is also worth making the point that, although secondary flows in the gas core are observed in boiler tubes, there is good experimental evidence that swirl and a consequent 'hydrocyclone' effect is *not* the mechanism that ensures that the fluid remains close to the tube walls.

Because annular flows arise in such a range of applications, it is hard to give any general rules for the conditions under which this regime pertains. In subsequent sections, however, 'typical' parameter values will be required, so it is as well to have some idea of the velocities involved. A two-phase flow regime map is presented on p. 316 of Wallis (1969) in which typical values for horizontal adiabatic annular flow are given as

$$\frac{G_\ell \lambda \psi}{G_g} \sim 20, \quad \frac{G_g}{\lambda} \sim 10^4$$

where

$$\lambda = \left[\left(\frac{\rho_g}{0.075} \right) \left(\frac{\rho_\ell}{62.3} \right) \right]^{1/2}, \quad \psi = \left[\left(\frac{73\mu_\ell}{\gamma} \right) \left(\frac{62.3}{\rho_\ell} \right)^2 \right]^{1/3}.$$

Here ρ_ℓ and ρ_g are the densities of the liquid and gas respectively (measured in lbs/ft³), μ_ℓ is the dynamic viscosity of the fluid (measured in lb/hr/ft), γ is the surface tension (measured in dynes/cm) and G_ℓ and G_g are the respective mass fluxes (lbs/hr/ft²). Using values for water and air at room temperature ($\gamma = 60$ dyne/cm, $\rho_g = 0.062$ lbs/ft³, $\rho_\ell = 62.3$ lbs/ft³ and $\mu_\ell = 2.5$ lb/hr/ft) gives $\lambda \sim 0.909$ and $\psi \sim 1.449$, and thus $u_g \sim 146613$ ft/hr = 12.41 m/sec and $u_\ell \sim 2216$ ft/hr = 0.188 m/sec.

In contrast to this, conditions for vertical cocurrent annular flow in a 1.25 inch diameter pipe atmospheric pressure are given on p. 10 of Wallis (1969). These suggest the onset of annular flow at a gas velocity of approximately 17.65 m/sec. Various values are also suggested for the existence of particular forms of disturbance to the flow. Roughly speaking, for a constant gas flow rate an increase in the liquid flow rate creates pulses, followed by disturbance waves with small ripple waves.

As a final example of annular two-phase flow, we consider vertical (non-adiabatic) flow in a vertical liquid metal fast breeder reactor boiler tube as discussed by Kane (1994). Here typical test section tubes have length 30m (of which 10m is occupied by two-phase flow) and diameter 12mm. Gas velocities of around 6 m/sec are normal, and liquid films of 0.5-1 mm are observed.

To summarise, therefore, we consider annular two-phase flows where the gas velocity is of the order of 10 m/sec and the fluid velocity is typically one or two orders of magnitude smaller. We assume that the annular flow regime persists so long as (a) the gas velocity is sufficiently high and (b) the fluid film does not become too thick.

3. Models for Annular Two-Phase Flow Disturbances

Before proposing a model for disturbances to annular two-phase flow, it is worth noting that we assume throughout that annular two-phase flow is the existing flow regime, and give no attention to the mechanisms that led to its establishment. Typically, annular flow is set up from a combination of slug/plug and churn turbulent flows, but the details of the establishment of the liquid film on the walls of the duct or tube are extremely complicated. (For tentative descriptions, see Wallis (1969).)

To set up a framework for a model, we assume that flow takes place in a two layers, a liquid layer $0 \leq y \leq h(x, t)$ which lies beneath a gas layer $L \geq y \geq h(x, t)$.

We non-dimensionalise according to $x = L\bar{x}$, $y = L\bar{y}$, $t = L/(U)\bar{t}$, $h = L\bar{h}$, $u_\ell = U\bar{u}_\ell$, $v_\ell = U\bar{v}_\ell$ and $p_\ell = \mu_\ell U/L\bar{p}_\ell$. The Navier-Stokes equations in two dimensions become (having dropped the bars for convenience)

$$Re[u_{\ell t} + u_\ell u_{\ell x} + v_\ell u_{\ell y}] = -p_{\ell x} + (u_{\ell xx} + u_{\ell yy}) - \frac{\sin \theta}{Fr^2} \quad (1)$$

$$Re[v_{\ell t} + u_\ell v_{\ell x} + v_\ell v_{\ell y}] = -p_{\ell y} + (v_{\ell xx} + v_{\ell yy}) + \frac{\cos \theta}{Fr^2} \quad (2)$$

$$u_{\ell x} + v_{\ell y} = 0, \quad (3)$$

in the liquid, and

$$Re\rho[u_{gt} + u_g u_{gx} + v_g u_{gy}] = -p_{gx} + \mu(u_{gxx} + u_{gyy}) - \rho \frac{\sin \theta}{Fr^2} \quad (4)$$

$$Re\rho[v_{gt} + u_g v_{gx} + v_g v_{gy}] = -p_{gy} + \mu(v_{gxx} + v_{gyy}) + \rho \frac{\cos \theta}{Fr^2} \quad (5)$$

$$u_{gx} + v_{gy} = 0, \quad (6)$$

in the gas.

Here the Reynolds number is taken to be $Re = LU/\nu_\ell$ where ν_ℓ is the kinematic viscosity of the liquid, and the Froude number is $Fr = \sqrt{\nu_\ell U/gL^2}$. In order to apply these equations to both the gas and liquid, we have introduced the viscosity ratio $\mu = \mu_g/\mu_\ell$ and the density ratio $\rho = \rho_g/\rho_\ell$. The angle θ has been introduced for simplicity to distinguish between three cases, namely $\theta = 0$ (the liquid layer on the top surface of a horizontal duct, hereinafter called "liquid on top"), $\theta = \pi$ (the liquid layer on the bottom surface of a horizontal duct, called "liquid on bottom") and $\theta = \pi/2$ (the liquid layer on a side wall of a vertical duct, called "liquid on side").

The boundary conditions at the interface between the gas and liquid are given by continuity of the velocity (the so-called “no-slip” condition),

$$u(x, h(x, t)^+) = u(x, h(x, t)^-) \quad (7)$$

$$v(x, h(x, t)^+) = v(x, h(x, t)^-) \quad (8)$$

and a jump condition for stress,

$$\mathbf{n} \cdot \left\{ -p\mathbf{I} + [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \right\}^- - \mathbf{n} \cdot \left\{ -p\mathbf{I} + \mu [\nabla \mathbf{v} + (\nabla \mathbf{v})^T] \right\}^+ = \bar{\gamma} \bar{H} \mathbf{n}, \quad (9)$$

where $\bar{\gamma}$ is the dimensionless surface tension coefficient, which is equal to $\gamma/\mu_\ell U$, and $\bar{H}$ is the dimensionless curvature.

The evolution of interface waves is described by using the kinematic condition that, on $y = h(x, t)$,

$$\frac{D}{Dt}(y - h(x, t)) = 0$$

and thus

$$h_t + u_\ell(x, h(x, t))h_x = v_\ell(x, h(x, t)). \quad (10)$$

In subsequent sections, several different modeling aspects of annular flow will be examined. In the first several models, the liquid layer is assumed to be thin and the gas is treated as inviscid, with the possibility of a (turbulent) boundary layer at the gas-liquid interface.

The two-layer Poiseuille model studies the stability of a flow generated by viscous effects to a perturbation where viscous effects are ignored.

3.1 Thin liquid layer model

In this section, we assume that the liquid layer adjoining the wall $y = 0$ is much thinner than the duct height, a small parameter ϵ being defined by the ratio of a typical h to L . In the liquid layer we non-dimensionalise and rescale according to $\bar{x} = \bar{x}$, $\bar{y} = \epsilon \bar{y}$, $\bar{t} = 1/\epsilon \bar{t}$, $\bar{h} = \epsilon \bar{h}$, $\bar{u}_\ell = \epsilon \bar{u}_\ell$, $\bar{v}_\ell = \epsilon^2 \bar{v}_\ell$ and $\bar{p}_\ell = 1/(\epsilon) \bar{p}_\ell$. These scalings not only take account of the relative thinness of the liquid layer, but also express the fact that, as discussed above, velocities in this layer are an order of magnitude smaller than in the gas core flow. The Navier-Stokes equations in two dimensions become (having again dropped the bars for convenience)

$$Re\epsilon[u_{tt} + u_\ell u_{\ell x} + v_\ell u_{\ell y}] = -\frac{1}{\epsilon^2} p_{\ell x} + u_{\ell xx} + \frac{1}{\epsilon^2} u_{\ell yy} - \frac{gL^2 \sin \theta}{\epsilon \nu_\ell U_\infty}$$

$$Re\epsilon^3[v_{\ell t} + u_{\ell}v_{\ell x} + v_{\ell}v_{\ell y}] = -\frac{1}{\epsilon^2}p_{\ell y} + \epsilon^2v_{\ell xx} + v_{\ell yy} + \frac{gL^2 \cos \theta}{\nu_{\ell}U_{\infty}}$$

$$u_{\ell x} + v_{\ell y} = 0.$$

Here $Re = LU_{\infty}/\nu_{\ell}$ where ν_{ℓ} is the kinematic viscosity of the liquid, and the angle θ has been introduced for simplicity to distinguish between three cases, namely $\theta = 0$ (the liquid layer on the top surface of a horizontal duct, hereinafter called liquid on top), $\theta = \pi$ (the liquid layer on the bottom surface of a horizontal duct, called liquid on bottom) and $\theta = \pi/2$ (the liquid layer on a side wall of a vertical duct, called liquid on side).

We now consider these equations in the limit of small ϵ . They become, to lowest order (in redimensionalised form)

$$p_{\ell x} = \mu_{\ell}u_{\ell yy} - \rho_{\ell}g \sin \theta \quad (11)$$

$$p_{\ell y} = \rho_{\ell}g \cos \theta \quad (12)$$

$$u_{\ell x} + v_{\ell y} = 0. \quad (13)$$

For horizontal flow ($\sin \theta = 0$), the conditions that these are the correct leading order equations are that

$$Re\epsilon^3 \ll 1, \quad gL\epsilon^2 \sim \nu_{\ell}U_{\infty}.$$

Using the representative values $\nu_{\ell} = 10^{-6}$ m²/sec, $\nu_g = 1.5 \times 10^{-5}$ m²/sec, this requires that $U_{\infty} \ll 10^{9/5}L^{1/5}$, so that for $U_{\infty} \sim 10$ m/sec (as suggested above), the tube width must exceed 10^{-4} m for the analysis to be valid; an eminently reasonable requirement.

For vertical flow, the conditions under which the leading order equations are as given above are slightly changed to

$$Re\epsilon^3 \ll 1, \quad gL\epsilon \sim \nu_{\ell}U_{\infty},$$

but once again it may easily be confirmed that these conditions do allow typical annular two-phase flows to be considered.

The equations (11)-(13) may easily be solved in terms of two arbitrary functions $P(x, t)$ and $A(x, t)$ which are to be determined, giving

$$p_{\ell} = \rho_{\ell}gy \cos \theta + P(x, t)$$

$$u_{\ell} = \frac{y^2}{2\mu_{\ell}}(P_x + \rho_{\ell}g \sin \theta) + yA(x, t)$$

$$v_{\ell} = -\frac{y^3P_{xx}}{6\mu_{\ell}} - \frac{y^2}{2}A_x.$$

Note that P is the pressure in the liquid over the hydrostatic, and that A is related the the stress at the interface by

$$\tau_i = \mu_\ell u_{\ell y} \Big|_{y=h} = h(P_x + \rho_\ell g \sin \theta) + \mu_\ell A(x, t)$$

we find that

$$A = \frac{\tau_i}{\mu_\ell} - \frac{h}{\mu_\ell}(P_x + \rho_\ell g \sin \theta).$$

The evolution equation for h is then

$$h_t = \left[\frac{h^3}{3\mu_\ell}(P_x + \rho_\ell g \sin \theta) - \frac{h^2}{2\mu_\ell}\tau_i \right]_x.$$

The model must now be closed by making suitable assumptions to obtain P and τ_i to couple the flow in the gas core to the problem.

3.2 A simple annular flow model

We begin by considering the simplest possible coupling model for the flow. Assuming that $p_\ell = 0$ and $\tau_i = \tau(h, h_x)$ on $y = h(x, t)$ where τ is a “known” function that characterises the influence of the gas stream on the liquid layer, we find that

$$P = -\rho_\ell g h \cos \theta, \quad A(x) = \frac{\tau}{\mu_\ell} + \frac{h\rho_\ell g}{\mu_\ell}[h_x \cos \theta - \sin \theta]$$

and thus

$$\begin{aligned} p_\ell &= \rho_\ell g \cos \theta (y - h) \\ u_\ell &= \frac{y^2}{2\mu_\ell}(-\rho_\ell g h_x \cos \theta + \rho_\ell g \sin \theta) + y \left(\frac{\tau}{\mu_\ell} + \frac{h\rho_\ell g}{\mu_\ell}[h_x \cos \theta - \sin \theta] \right) \\ v_\ell &= \frac{y^3}{6\mu_\ell}\rho_\ell g h_{xx} \cos \theta - \frac{y^2}{2} \left[\frac{\tau}{\mu_\ell} + \frac{\rho_\ell g h}{\mu_\ell}(h_x \cos \theta - \sin \theta) \right]_x. \end{aligned}$$

The evolution equation for $h(x, t)$ is thus

$$\begin{aligned} h_t + \frac{h^2 h_x^2}{\mu_\ell} \rho_\ell g \cos \theta + h h_x \frac{\tau}{\mu_\ell} + \frac{h^2}{2\mu_\ell} (\partial_h \tau h_x) = \\ - \frac{h^3}{3\mu_\ell} \rho_\ell g h_{xx} \cos \theta - \frac{h^2}{2\mu_\ell} h_{xx} (\partial_{h_x} \tau) + \frac{h^2 h_x}{\mu_\ell} \rho_\ell g \sin \theta \end{aligned} \quad (14)$$

Three separate cases may be identified: (i) $\theta = 0$ (liquid on top). Here the first term on the right hand side of (14) is of the nature of a backward diffusion term. As expected therefore,

gravity destabilises the film. The only hope of changing the sign of this term would be if $\partial_{h_x}\tau$ was sufficiently negative. (ii) $\theta = \pi$ (liquid on bottom). Here gravity stabilises the interface; for $\partial_{h_x}\tau$ positive and sufficiently large, however, instability could still be created. (iii) $\theta = \pi/2$ (liquid on side). The stability now depends solely upon the sign of $\partial_{h_x}\tau$. This

qualitative argument suggests that the dynamics of the gas flow must be included in any model that will be capable of predicting a “stable” liquid film in most cases. This was carried out in two parts, first, by including a pressure fluctuation term, and second, by including a boundary layer drag between the gas and the liquid.

3.2.1 Effects of gas core flow

In order to try to quantify more clearly the effects of the gas core flow, we consider the gas flow to be inviscid and irrotational; the pressure is assumed to jump in proportion to the surface tension times curvature at the gas/fluid interface, whilst the shear stress resulting from the viscous forces exerted by the gas on the fluid is incorporated into the model by specifying $\mu_\ell u_{\ell y} = \tau_i$ at $y = h(x, t)$.

Thin aerofoil theory Because there was some confusion over the signs of various quantities during the discussion that took place at the workshop, we derive the thin aerofoil pressure relationship from first principles. In the gas flow, we have, from Bernoulli’s equation,

$$p_g + \frac{1}{2}\rho_g v_g^2 = p_\infty,$$

where p_∞ is a constant. Assuming that a boundary perturbation of $O(\epsilon)$ produces an $O(\epsilon)$ change to the gas flow, we seek an expression for the stream function of the disturbance potential in the gas flow of the form

$$\psi_g(x, y, t) = U_g y + \frac{\epsilon U_g}{\pi} \int_{-\infty}^{\infty} g(\xi, t) \tan^{-1} \left(\frac{y}{x - \xi} \right) d\xi + o(\epsilon)$$

where g is to be determined. We note that, according to the definition of this stream function,

$$u_g = U_g + \frac{\epsilon U_g}{\pi} \int_{-\infty}^{\infty} g(\xi, t) \frac{x - \xi}{(x - \xi)^2 + y^2} d\xi \rightarrow U_g + \frac{\epsilon U_g}{\pi} \int_{-\infty}^{\infty} \frac{g(\xi)}{x - \xi} d\xi \quad (y \rightarrow 0)$$

and

$$v_g = \frac{\epsilon U_g}{\pi} \int_{-\infty}^{\infty} g(\xi, t) \frac{y}{(x - \xi)^2 + y^2} d\xi \rightarrow \epsilon U_g g(x) \quad (y \rightarrow 0),$$

where U_g is the streaming velocity of the gas. The kinematic boundary condition in the outer flow is

$$h_t + u_g(x, h(x, t))h_x = v_g(x, h(x, t)). \quad (15)$$

Using the obvious scalings $x = L\bar{x}$, $t = L/(U_\infty \epsilon)\bar{t}$, $h = \epsilon L\bar{h}$, $u_g = U_g\bar{u}_g$ and $v_g = \epsilon U_g\bar{v}_g$ shows that the first term of (15) is negligible, and therefore to lowest order $\bar{u}_g\bar{h}_{\bar{x}} = \bar{v}_g$ on $\bar{y} = \bar{h}(\bar{x}, \bar{t})$. Thus $g(\bar{x}) = \bar{h}_{\bar{x}}$ and, upon using

$$p_g = p_\infty - \frac{1}{2}\rho_g[u_g^2 + v_g^2]$$

we find that (in dimensional variables)

$$p_g = p_\infty - \frac{1}{2}\rho_g U_g^2 + \frac{\rho_g U_g^2}{\pi} \int_{-\infty}^{\infty} \frac{h_x(\xi, t)}{\xi - x} d\xi. \quad (16)$$

For simplicity, we choose $p_\infty = \frac{1}{2}\rho_g U_g^2$. Then we have

$$p_g = \frac{\rho_g U_g^2}{\pi} \int_{-\infty}^{\infty} \frac{h_x(\xi, t)}{\xi - x} d\xi. \quad (17)$$

Boundary layer drag The interfacial stress represents the force per unit area on the liquid exerted by the gas. The model for the gas flow assumes that the gas is moving at speed U_g in the x -direction. A “standard” interfacial drag model gives

$$\tau_i = f_i \rho_g (U_g - u_i)^2, \quad (18)$$

where $u_i = u_\ell(x, h)$, and f_i is a friction factor. Wallis (1969) suggests a value of

$$f_i = 0.0025(1 + 300\epsilon)$$

which gives a value of about 0.025 for $\epsilon = 1/30$, a representative value.

The stress is coupled to the motion through the condition that

$$u_i = u_\ell(x, h) = \frac{h^2}{2\mu_\ell}(P_x + \rho_\ell g \sin \theta) - \frac{h}{\mu_\ell} \tau_i.$$

This equation, together with (18), give the interface velocity implicitly. This quadratic equation can be solved (assuming that f_i is constant) to get

$$u_i = U_g - \frac{\mu_\ell}{2hf_i\rho_g} \left(-1 + \sqrt{1 + \frac{4hf_i\rho_g U_g}{\mu_\ell} + \frac{2h^3 f_i \rho_g}{\mu_\ell^2} (P_x + \rho_\ell g \sin \theta)} \right)$$

Note that the interfacial stress is given by

$$\tau_i = \frac{\mu_\ell^2}{4h^2 f_i \rho_g} \left(-1 + \sqrt{1 + \frac{4hf_i \rho_g U_g}{\mu_\ell} + \frac{2h^3 f_i \rho_g}{\mu_\ell^2} (P_x + \rho_\ell g \sin \theta)} \right)^2. \quad (19)$$

Note further that if we assume that

$$1 \gg \frac{4hf_i \rho_g U_g}{\mu_\ell} \gg \frac{2h^3 f_i \rho_g}{\mu_\ell^2} (P_x + \rho_\ell g \sin \theta), \quad (20)$$

we have

$$\tau_i \sim f_i \rho_g U_g^2 + \frac{f_i \rho_g U_g h^2}{\mu_\ell} (P_x + \rho_\ell g \sin \theta) \quad (21)$$

3.2.2 Lubrication/Thin Aerofoil Model

We may now proceed to solve the governing equations with the further assumption that the flow does not interact with the interfacial stress, viz.,

$$\tau_i \sim f_i \rho_g U_g^2. \quad (22)$$

To determine P we have

$$p_\ell = \rho_\ell g y \cos \theta + P$$

and, on $y = h(x, t)$,

$$p_\ell = p_g - \gamma h_{xx}$$

Using (17), we finally find that the system to be solved for $h(x, t)$ is

$$h_t = \left[\frac{h^3}{3\mu_\ell} (P_x + \rho_\ell g \sin \theta) - \frac{f_i \rho_g U_g^2 h^2}{2\mu_\ell} \right]_x, \quad (23)$$

$$P_x = -\gamma h_{xxx} - \rho_\ell g h_x \cos \theta + \left[\frac{\rho_g U_g^2}{\pi} \int_{-\infty}^{\infty} \frac{h_x(\xi, t)}{\xi - x} d\xi \right]_x. \quad (24)$$

Some discussion of (23) and (24) is apposite. The equations may be thought of as a marriage between lubrication and thin aerofoil theory. Such equations have been studied before; King & Tuck (1993) consider the support against gravity of a thin liquid layer on a plane wall in an upward flow of air, citing the example of rain drops on the windscreen of a car travelling at high speed. They considered steady flows only, and the drops were of finite extent. The dislodging by dynamic pressure forces of a drop against surface tension was studied by Durbin (1988). Once again, only steady flows were considered but downstream

of the (finite extent) drop a wake with an unknown separation point was assumed to exist. In both these cases numerical solutions to a steady version of (23) and (24) were computed. Equations similar to (23) and (24) were derived by Fitt et al. (1995) in a study of crack propagation in a geothermal energy reservoir. In this case, the point at issue was the crack spreading mechanism and similarity and crack tip boundary layer solutions were determined. Finally, the study of King, Tuck & Vandenbroeck (1993) concerned steady solutions to a model closely related to that presented above for waves on a wind-blown thin inclined layer of viscous fluid. They found that steady periodic waves existed and that there was a long-wave limit of a solitary waveform.

Steady Solutions Before considering the full unsteady model, we discuss some steady solutions of (23) and (24). When time-dependence is ignored, (23) may immediately be integrated. Identifying the constant of integration with the fluid flux $-Q$ in the liquid layer we find that

$$Q = \frac{h^2}{4\mu_\ell} f_i \rho_g U_g^2 - \frac{h^3}{3\mu_\ell} (P_x + \rho_\ell g \sin \theta),$$

$$P_x = -\gamma h_{xxx} - \rho_\ell g h_x \cos \theta + p_{gx}.$$

With $h_t = 0$, the flux Q is constant, and the final equations for h is

$$-\gamma h_{xxx} - g \rho_\ell h_x \cos \theta + p_{gx} = \frac{3 f_i \rho_g U_g^2}{4h} - \frac{3Q\mu_\ell}{h^3} - g \rho_\ell \sin \theta.$$

This is essentially a generalisation of the equation studied by King et al. (1993), and similar methods to those examined in that paper could be used for analysis; as usual, for a given Q constant height solutions are determined by

$$\frac{f_i \rho_g U_g^2}{4} = \frac{Q\mu_\ell}{h^2}.$$

Linear stability To investigate the stability of interface waves in annular flow, we first carry out a standard linear stability analysis upon (23) and (24). Setting

$$h = h_0 + h_1 e^{i(kx - \omega t)}$$

where h_0 is a constant and $h_1 \ll 1$, and using the fact that

$$\frac{1}{\pi} \int_{-\infty}^{\infty} \frac{(e^{ik\xi})'}{\xi - x} d\xi = -|k| e^{ikx},$$

we find that the dispersion relation for (23) and (24) is

$$-i\omega = \frac{h_0^3 k^2}{3\mu_\ell} [-\gamma k^2 + \rho_\ell g \cos \theta + |k| \rho_g U_g^2] + \frac{ik h_0}{\mu_\ell} (h_0 g \rho_\ell \sin \theta - f_i \rho_g U_g^2) \quad (25)$$

and we may now consider the three obvious cases:

(i) $\theta = 0$ (liquid on top). Here both gravity and the dynamic pressure forces are destabilizing. For stability we require

$$\gamma k^2 > \rho_\ell g + |k| \rho_g U_g^2$$

and, taking $k > 0$ without loss of generality, the critical wavenumber k_c is therefore given by

$$k_c = \frac{\rho_g U_g^2 + \sqrt{\rho_g^2 U_g^4 + 4\gamma \rho_\ell g}}{2\gamma}.$$

Using the typical parameter values quoted above, this gives critical wave numbers and corresponding wavelengths λ of $k_c = 417$, $\lambda = 15\text{mm}$ ($U_g = 1$); $k_c = 667$, $\lambda = 9.4\text{mm}$ ($U_g = 5$) and $k_c = 1761$, $\lambda = 3.6\text{mm}$ ($U_g = 10$). The most unstable wave has wavenumber $\rho_g U_g^2 / (2\gamma)$ and hence a wavelength of about $8.33 U_g^2$. The wave speed is given by

$$c = -\frac{\text{Im}(\omega)}{k} = \frac{h_0 f_i \rho_g U_g^2}{\mu_\ell}$$

and waves are therefore not dispersive, waves of different wave number travelling at identical speeds. For practical purposes, c may be tentatively estimated as follows: Assuming from our earlier analysis that $gL\epsilon^2 \sim \nu_\ell U_g$, we have, with the usual parameter values,

$$c \sim 1.15 \times 10^{-2} U_g^{5/2}.$$

Thus for $U_g \sim 10$ the wave speed is likely to be between 3 and 4 m/sec. It must be emphasised that these rough calculations should be repeated with other correlations available in the literature until some sort of consensus is achieved.

(ii) $\theta = \pi$ (liquid on bottom). Here the gravity stabilises the flow and for stability we require

$$\gamma k^2 + \rho_\ell g > k \rho_g U_g^2.$$

Although we have stability for large and small k , the maximum value of ω is realised when $k = \rho_g U_g^2 / (2\gamma)$. At this values of k we have

$$\omega = \frac{1}{4\gamma} (\rho_g^2 U_g^4 - 4\gamma \rho_\ell g)$$

and for typical values this is likely to exceed zero, so that it is likely that there will still be unstable wave numbers. Waves still travel at a constant speed identical to case (i). Investigations are still continuing in this case.

(iii) $\theta = \pi/2$ (liquid on side). Here the stability criterion is simply that $\gamma k > \rho_g U_g^2$, so long waves are always unstable. Further analysis is also possible in this case, but has not been considered yet.

Effect of Gas Boundary Layer The condition for the model presented in eq (20) is difficult to meet for the parameters discussed in this report, with

$$\frac{4hf_i\rho_g U_g}{\mu_l} = O(1) \quad (26)$$

for the values presented here. However, if we assume that the liquid layer is turbulent, we can use values of the eddy viscosity to ascertain the validity of the inequality (20). If the eddy viscosity is more than an order of magnitude bigger than the laminar viscosity, then the conditions are met.

Next, We proceed to analyze the governing equations, accounting for the interaction of the flow with the interfacial stress. We assume

$$\tau_i \sim f_i \rho_g U_g^2 + \frac{f_i \rho_g U_g h}{\mu_l} (P_x + \rho_l g \sin \theta) \quad (27)$$

Again, we have

$$p_l = \rho_l g y \cos \theta + P$$

and, on $y = h(x, t)$,

$$p_l = p_g - \gamma h_{xx}$$

Using (17), we finally find that the system to be solved for $h(x, t)$ is

$$h_t = \left[\frac{h^3}{3\mu_l} \left(1 - \frac{3f_i \rho_g U_g h_0}{2\mu_l} \right) (P_x + \rho_l g \sin \theta) - \frac{f_i \rho_g U_g^2 h^2}{2\mu_l} \right]_x, \quad (28)$$

$$P_x = -\gamma h_{xxx} - \rho_l g h_x \cos \theta + \left[\frac{\rho_g U_g^2}{\pi} \int_{-\infty}^{\infty} \frac{h_x(\xi, t)}{\xi - x} d\xi \right]_x. \quad (29)$$

Note that the constant height solution is unchanged.

Linear stability The dispersion relation becomes

$$-i\omega = \frac{h_0^3 k^2}{3\mu_\ell} \left(1 - \frac{3f_i \rho_g U_g h_0}{2\mu_i} \right) [-\gamma k^2 + \rho_\ell g \cos \theta + |k| \rho_g U_g^2] + \frac{ik h_0}{\mu_\ell} (h_0 g \rho_\ell \sin \theta - f_i \rho_g U_g^2) \quad (30)$$

The effect of the boundary layer drag is to decrease the growth rate when the gas velocity increases. Indeed, when the gas velocity reaches

$$U_g = \frac{2\mu_i}{3f_i \rho_g h_0}$$

the growth rate changes sign, and for larger gas velocities, the flat interface is stable. This criterion implies that if annular flow occurs at 17.65 m/sec, and the laminar viscosity is used, a stable film of thickness $h_0 = 0.157$ cm can exist at the top of the duct. However, we see that this contradicts the assumptions used to arrive at this model. If we instead assume that the viscosity represents a turbulent viscosity, of an order of magnitude bigger than laminar, the maximum thickness of the stable film increases by an order of magnitude.

3.3 Stability of Poiseuille Flow

Consider flow in three layers as depicted in figure 1. For this analysis, we shall use a different coordinate system, and take $\theta = \pi/2$. For plane Poiseuille flow with interfaces at $y = \pm H$, $v = 0$, p_x is constant and $u_0(y)$ is given by

$$u_0(y) = u_\ell = a_1(1 - y^2) \quad \text{for } h_0^2 \leq y^2 \leq 1 \quad (31)$$

$$u_0(y) = u_g = a_1(1 - h_0^2) + a_2(h_0^2 - y^2) \quad \text{for } 0 \leq y^2 \leq h_0^2 \quad (32)$$

and the pressure is given by

$$p_0 = \begin{cases} cx + Fr^{-2}(1 - y) & \text{for } h_0 \leq y \leq 1 \\ cx + Fr^{-2}(1 - h_0) + \rho Fr^{-2}(h_0 - y) & \text{for } 0 \leq y \leq h_0. \end{cases} \quad (33)$$

Without loss of generality, we choose the velocity scale so that $u_0(h_0) = 1$. This requires $c = -\frac{1}{1-h_0^2}$, so that $a_1 = \frac{1}{2(1-h_0^2)}$ and $a_2 = \frac{1}{2\mu(1-h_0^2)}$.

Introduce small waves along the upper interface such that

$$u(x, y, t) = u_0(y) + u_1(x, y, t) \quad (34)$$

$$v(x, y, t) = v_1(x, y, t) \quad (35)$$

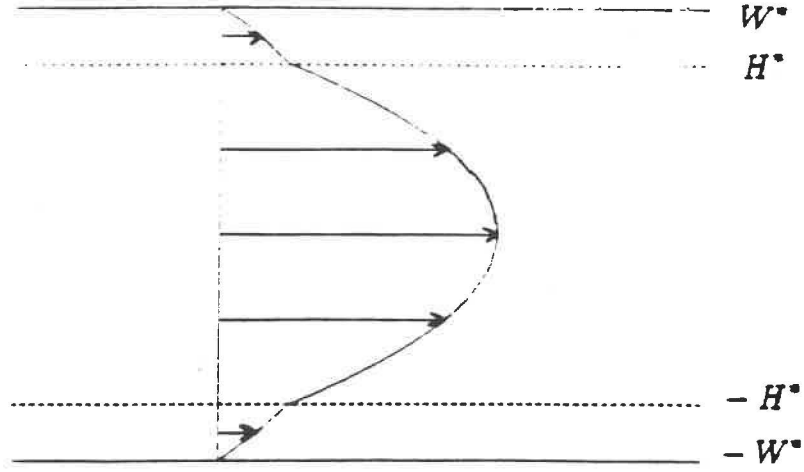


Figure 1: Layered Poiseuille flow between horizontal plates.

$$p(x, y, t) = p_0 + p_1(x, y, t) \quad (36)$$

$$h(x, t) = h_0 + h_1 e^{i(kx - \omega t)} \quad (37)$$

The kinematic condition (10) implies

$$v_1(x, y, t) = \nu(y) e^{i(kx - \omega t)} \quad (38)$$

with

$$\nu(h_0) = i h_1 (-\omega + k). \quad (39)$$

The continuity equation (3) implies

$$u_1(x, y, t) = -\frac{\nu'(y)}{ik} e^{i(kx - \omega t)}. \quad (40)$$

Let

$$p_1(x, y, t) = P_1(y) e^{i(kx - \omega t)} \quad (41)$$

Assume that viscous effects in the perturbed flow can be ignored. Then, equations (1) and (4) become respectively

$$Re \left(\frac{\omega}{k} - u_0 \right) \nu' + \nu u_0' = -ik P_1(y) + (1 - \rho) \frac{1}{Fr^2} i h_1 k \quad (42)$$

for $0 \leq y \leq h_0$, and

$$\text{Re} \rho \left(\frac{\omega}{k} - u_0 \right) \nu' + \nu u_0' = -ikP_1(y) \quad (43)$$

for $h_0 \leq y \leq 1$. Furthermore, equations (2) and (5) become

$$\text{Re}(-i\omega\nu + u_0 ik\nu) = -P_1'. \quad (44)$$

Eliminating P_1' from above yields Rayleigh's equation

$$\left(\frac{\omega}{k} - u_0 \right) \nu'' + \left(u_0'' - k^2 \left(\frac{\omega}{k} - u_0 \right) \right) \nu = 0. \quad (45)$$

The problem then reduces to solving equation (45) for ν in both layers with the boundary conditions that ν vanishes at the upper wall, $y = 1$, and somewhere in the lower layer (away from the interface). In addition, the vertical velocity, ν , and pressure, P_1 , must match at the interface. Since the perturbed flow is assumed to be inviscid, the horizontal velocity is allowed to slip at the interface.

3.3.1 Stability Analysis

Since the top layer is thin relative to the bottom layer, define

$$\sigma = 1 - h_0 \quad \text{where} \quad \sigma \ll 1. \quad (46)$$

Equation (45) is a linear equation in the unknown function ν with nonconstant coefficients. It can be solved simply for the two cases $|u_0''| \gg |k^2(\omega/k - u_0)|$ or $|u_0''| \ll |k^2(\omega/k - u_0)|$.

Case $|u_0''| \gg |k^2(\frac{\omega}{k} - u_0)|$: This limit corresponds to $k \ll 1$ or waves which are long relative to the height of the lower layer. Then, equation (45) can be approximated by

$$\left(\frac{\omega}{k} - u_0 \right) \nu'' + u_0'' \nu = 0 \quad (47)$$

which can be integrated once by parts to yield

$$\left(\frac{\omega}{k} - u_0 \right) \nu' + u_0' \nu = A. \quad (48)$$

Equation (48) is a first order linear equation in the unknown f and can be evaluated using an integrating factor when $(\omega/k - u_0) \neq 0$. Since u_0 is zero on the upper boundary and very large at $y = 0$, the real part of $(\omega/k - u_0)$ is expected to vanish somewhere in between. Proceeding naively, ν is found in the upper layer to be

$$\nu = (\alpha^2 - y^2) \int_{\tilde{y}=y}^1 \frac{2\sigma A_1}{(\alpha^2 - \tilde{y}^2)^2} d\tilde{y} \quad (49)$$

for

$$\alpha^2 = 1 + 2\sigma\left(1 - \frac{\omega}{k}\right). \quad (50)$$

Note ν vanishes at the upper boundary by construction. However, the integral in (49) does not exist if the denominator vanishes within the interval of integration. Since $1 \leq y \leq 1 + \alpha$ in the upper layer, the denominator will vanish for some y if ω/k is real and positive. Evaluating (49) asymptotically for $y > \text{Re}(\alpha)^+$ and $\sigma \ll 1$ gives the alternate expression

$$\nu_1 \sim -A_1\sigma \frac{(W - y)}{(W - \alpha)} \text{ as } \sigma \rightarrow 0. \quad (51)$$

Expression (51) exists for all y in the top layer and satisfies (48) asymptotically for small σ (with $A = A_1$), unless $\omega/k = u_0$ in the upper layer.

The solution for ν in the bottom layer is

$$\nu_2 = (\beta^2 - y^2) \int_{\tilde{y}=L}^y \frac{-2\mu\sigma A_2}{(\beta^2 - \tilde{y}^2)^2} d\tilde{y} \quad (52)$$

where

$$\beta^2 = \frac{(1 - \omega/k)(2\sigma + \sigma^2)}{D} + 1 \quad (53)$$

and L is some arbitrary lower limit where ν_2 vanishes. To leading order, the exact location of L is not important as long as it is away from the interface at $y = 1 - h_0$.

In nondimensional form, the kinematic condition (39) is equivalent to

$$\nu(h_0) = ih_1k\left(1 - \frac{\omega}{k}\right). \quad (54)$$

letting

$$\tilde{\omega} = \left(1 - \frac{\omega}{k}\right), \quad (55)$$

and substituting equations (51) and (52) into (54) yields

$$A_1 \sim \frac{ih_1k}{\sigma} \tilde{\omega}(\tilde{\omega} - 1) \text{ as } \sigma \rightarrow 0, \quad (56)$$

and

$$A_2 \sim \frac{-ih_1k}{\mu\sigma} \tilde{\omega} \text{ as } \sigma \rightarrow 0. \quad (57)$$

Matching the pressure at the interface implies

$$A_1 = \rho A_2 - (1 - \rho)ih_1k \frac{1}{Fr^2}. \quad (58)$$

Substituting for A_1 and A_2 from (56) and (57) yields

$$\tilde{\omega}(\tilde{\omega} - 1) = -\frac{\rho}{\mu}\tilde{\omega} - (1 - \rho)\sigma\frac{1}{Fr^2}. \quad (59)$$

Solving for $\tilde{\omega}$ from (59) gives

$$\tilde{\omega} = \left(1 - \frac{\omega}{k}\right) = \frac{(1 - \frac{\rho}{\mu}) \pm \sqrt{(1 - \frac{\rho}{\mu})^2 - 4(1 - \rho)\sigma\frac{1}{Fr^2}}}{2}. \quad (60)$$

The wave does not grow if the discriminant in (60) is nonnegative. Thus, the flat interface is stable to waves of short wavenumber ($|u_0''| \gg |k^2(\frac{\omega}{k} - u_0)|$) if

$$\left(1 - \frac{\nu_g}{\nu_\ell}\right)^2 \geq 4\left(1 - \frac{\rho_g}{\rho_\ell}\right)\left(\frac{W - H}{H}\right)\frac{gH}{U^2(H)}. \quad (61)$$

where H is the undisturbed height of the gas layer. Note that in this approximation, to leading order, the waves are not dispersive; i.e., ω/k is constant. Note also that for all other parameters held fixed, the growth rate decreases as the velocity at the interface increases or the ratio of the top layer height to the bottom layer height decreases.

Case $|u_0''| \ll |k^2(\frac{\omega}{k} - u_0)|$: This limit corresponds to $k \gg 1$ or waves short compared to the height of the bottom. Then equation (45) can be approximated by

$$\nu'' - k^2\nu = 0 \quad (62)$$

which has solutions

$$\nu_1 = A_1 \sinh k(1 - y) \quad (63)$$

$$\nu_2 = A_2 \sinh k(y - L) \quad (64)$$

where again to leading order L does not enter into the problem as long as L is away from $1 - h_0$. The kinematic condition (54) implies

$$A_1 = \frac{ih_1 k \tilde{\omega}}{\sinh k\sigma} \quad (65)$$

$$A_2 = \frac{ih_1 k \tilde{\omega}}{\sinh k(h_0 - L)}. \quad (66)$$

Matching the pressure at the interface yields

$$\tilde{\omega}^2 \left(k \frac{\cosh k\sigma}{\sinh k\sigma} + \rho k \right) + \tilde{\omega} \left(\frac{\rho - 1}{\sigma} \right) + (1 - \rho) \frac{1}{Fr^2} = 0 \quad (67)$$

or

$$\tilde{\omega} = 1 - \frac{\omega}{k} = \frac{\left(\frac{1-\rho}{\sigma}\right) \pm \sqrt{\left(\frac{\rho}{\sigma}\right)^2 - 4\left(k \frac{\cosh k\sigma}{\sinh k\sigma} + \rho k\right)(1-\rho)\frac{1}{Fr^2}}}{2\left(k \frac{\cosh k\sigma}{\sinh k\sigma} + \rho k\right)}. \quad (68)$$

Thus, the flat interface is stable to waves of wavenumber k if

$$\left(1 - \frac{\nu_g}{\nu_\ell}\right)^2 \geq 4kH \left(\frac{\cosh k(W-H)}{\sinh k(W-H)} + \frac{\rho_g}{\rho_\ell}\right) \left(1 - \frac{\rho_g}{\rho_\ell}\right) g \frac{H}{U^2(H)} \left(\frac{W-H}{H}\right)^2. \quad (69)$$

Note that short waves are dispersive. Moreover, the flat interface is unstable to waves that are sufficiently short.

4. Discussion and Conclusions

Several different modeling aspects of annular flow have been examined. The situation with liquid above the gas (unstable density stratification) is unstable for the simplest sorts of models; the analysis focussed on the stability of a flat interface to small disturbances.

In the first several models, the liquid layer was assumed to be thin and the gas was treated as inviscid, with the possibility of a (turbulent) boundary layer at the gas-liquid interface. This model predicts that the stability depends on the model taken for the boundary layer stress. The range of parameters where stability was found was not satisfying—the liquid layer was assumed to be turbulent, so that the eddy viscosity could be used to justify the model that predicted stability of the film. These assumptions seem to contradict each other.

Although it seems that a fully nonlinear stability analysis would be required to determine the fate of long waves, the fact that, according to linear theory, the fastest growing wave has such a long wave length suggests that for tubes of modest sizes it might not be observed. A tentative explanation for the persistence of a film on the top wall of a horizontal duct may therefore be simply that unstable modes simply leave the boiler tube before they have had a chance to grow very much.

One of the results of this analysis is that surface tension plays a role in stabilizing short waves. Consequently, the wavenumber where stability changes is a function of surface tension. Its influence and the fact that instabilities may pass out of the annular flow region too quickly to manifest themselves in any noticeable way seem to be the only mechanisms that have been identified for the maintenance of a true annular flow regime in a horizontal tube.

The two-layer Poiseuille model studies the stability of a flow generated by viscous effects to an inviscid perturbation. This model also predicts that the flat interface is always unstable, since for given flow parameters, there is always a wave number range for which the linear

stability calculation shows that the waves grow. It predicts that sufficiently short waves grow, and that the shorter waves grow faster. Long waves do not grow. Moreover, long waves are only weakly dispersive. If all other parameters are held fixed, the growth rate decreases as the velocity at the interface increases or the ratio of the top layer height to the bottom layer height decreases. Therefore, if we postulate an equilibration of the short waves (this cannot be predicted by linear theory) this analysis predicts that the interface will appear to be wavy, with an equilibrated short wave pattern on it, which is not unstable to long waves. The model does not incorporate the effects due to the difference in molecular viscosity when two fluids, for example, are of the same density. It is expected that these effects must be accounted for by using a viscous perturbed wave.

In the work presented above, no account has been taken of drop entrainment or deposition. Although much work has been carried out to investigate the role of drop formation by undercutting of interface waves and subsequent redeposition in annular two-phase flows (see, for example Wallis (1969)) it seems rather unlikely that the drops are solely responsible for the maintenance of the liquid layer.

The implications on the constitutive equations for the average fields is far from clear. As so often in the field of multiphase flow, trying to use a single two-phase flow equipped with a variety of added terms to model a wide range of flow regimes is fraught with problems.

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