Anthropometry Formulas
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\[
F = ma, \quad \frac{dv}{dt} = \frac{F}{m} \quad \tau = Ia, \quad \frac{d\omega}{dt} = \frac{\tau}{I}
\]

Segment Dimensions

Length of body segments is often obtainable by direct measurement. If not, the segment lengths can be estimated from the subject height using the proportions shown in Fig. 4.1, Winter, 4th ed., 2009.

Figure 4.1  Body segment lengths expressed as a fraction of body height $H$.

Density Estimation

\[
d = 0.69 + 0.9c
\]

\[
c = \frac{h}{w^{1/3}}
\]

Segment Densities

Distal extremities are bonier, and therefore more dense, than proximal parts of extremities. Typical segment densities for an “average” person (i.e. with density approx. 1.066 kg/l) are shown in right hand column of Table 4.1 in Winter, 4th ed., 2009. Density of these segments scales roughly with body density. See straight lines in Fig. 4.2. Each line is a segment density (y) as a function of whole body density (x). Careful measurement of three of these lines suggests that the dimensionless slopes are 1.75 for hands, 0.95 for foot, and 0.80 for thigh.

Suggested procedure for estimating segment density: estimate the subject whole body density using equation above (requires subject height and weight). Use the relevant line in Fig. 4.2 to determine the segment density that goes with that whole body density. If the segment is not shown in Fig. 4.2, then assume that the segment density would be the number in the right hand column of Table 4.1 for that segment, if whole body density is 1.066 kg/l, and assume the slope of the line relating segment density to whole body density has a slope of 1 (which is in the middle of the different slopes shown in Fig. 4.2). In other words, add to or subtract from the segment density the amount by which the whole body density exceeds or is less than 1.066 kg/l.

Example:

What are the densities of the forearm and of the head and neck for a person with h=1.68m, w=65 kg?

c=h/(w^{1/3}) = 0.4104 therefore total body density d = 0.69 +0.9 c = 1.059 kg/l.

Forearm appears in Figure 4.2. The figure indicates that when body density=1.059, forearm density = 1.12 kg/l.

Head & neck density does not appear in Fig. 4.2 so we look at Table 4.1, which indicates that “typical” head & neck density = 1.11 kg/l. Typical means for a person with whole body density = 1.066 kg/l. Our subject’s density is 1.059 kg/l.

Total body density – typical body density = 1.059-1.066 = -0.007 kg/l.

Therefore head & neck density = 1.11 – 0.007 = 1.103 kg/l.

Segment Mass

If you have scans or other detailed information about segment geometry, use:

\[ m_{tot} = \sum_i m_i = \delta \sum_i V_i \]

where \( \delta \) = segment density, and \( V_i \) = volume of the \( i^{th} \) “piece” of the segment. If the pieces are segments with thickness \( \Delta x \) and area \( A_i \), then \( V_i = \Delta x A_i \), and

\[ m_{tot} = \delta \Delta x \sum_i A_i \]

If you don’t have detailed information, estimate segment mass using column 3 of Table 4.1 in Winter, 4th ed., 2009.
Segment Center of Mass

If you have detailed geometric information (such as CT scans or circumferential measurements at different locations), use the equation

\[ x_{cm} = \frac{\sum_i m_i x_i}{m_{tot}} \]

where \( m_i \) is the mass of the \( i \)th slice through the segment. This is a slice perpendicular to the x axis (i.e. a slice parallel to the y-z plane), centered at position \( x_i \) along the x axis. If the segment density is \( \delta \), and the slice thickness is \( \Delta x \), and the area of the slice is \( A_i \), then \( m_i = \delta \cdot \Delta x \cdot A_i \), and the location of the center of mass, along the x axis, is

\[ x_{cm} = \frac{\delta \Delta x \sum_i A_i x_i}{m_{tot}} \]

For limb segments, it is usually convenient to choose a coordinate system in which one axis is the long axis of the limb, with its origin at the proximal or distal end. If the long axis is the x axis, then the c.m. in the y and z directions will be quite close to zero (i.e. along the long axis), for a subject without deformity. If deformity is present, or for non-limb segments, or when higher accuracy is desired, it may be necessary to compute the center of mass location with respect to the other axes also:

\[ y_{cm} = \frac{\sum_i m_i y_i}{m_{tot}} = \frac{\delta \Delta y \sum_i A_i y_i}{m_{tot}} \]

and likewise for \( z_{cm} \). Note that the \( A_i \)'s in the preceding equation are different from the earlier \( A_i \)'s: they are the areas of slices perpendicular to the y axis, whereas earlier the \( A_i \)'s were areas of slices perpendicular to the x axis.

If you don’t have detailed segment geometry information, use Table 4.1 in Winter, 4th ed., 2009, which gives c.m. as a fraction of segment length, from proximal end and from distal end. Estimate segment lengths by using Fig. 4.1 and subject height.

Center of Mass of Multiple Segments

Once the locations of the centers of mass of several segments are known, the location of the overall c.m. is computed as follows:

\[ \bar{x} = \frac{\sum_{i=1}^{N} m_i x_i}{\sum_{i=1}^{N} m_i}, \quad \bar{y} = \frac{\sum_{i=1}^{N} m_i y_i}{\sum_{i=1}^{N} m_i}, \quad \bar{z} = \frac{\sum_{i=1}^{N} m_i z_i}{\sum_{i=1}^{N} m_i} \]

where \( N \) = the number of segments, and where the center of mass of segment \( i \) is at \( (x_i, y_i, z_i) \). You may need to use trigonometry to determine the segment center of mass locations.

Example 1: Table shows the mass and segment center of mass location for three segments. Determine \( \bar{x} \) and \( \bar{y} \).

<table>
<thead>
<tr>
<th>Segment</th>
<th>Mass ( m_i ) (kg)</th>
<th>( x_i ) (m)</th>
<th>( y_i ) (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.5</td>
<td>0.121</td>
<td>0.427</td>
</tr>
</tbody>
</table>
Since there are 3 segments, N=3.

\[
\bar{x} = \frac{\sum_{i=1}^{3} m_i x_i}{\sum_{i=1}^{3} m_i} = \frac{m_1 x_1 + m_2 x_2 + m_3 x_3}{m_1 + m_2 + m_3} = \frac{(0.6655 - 0.2940 + 0.0396) kg \cdot m}{(5.5 + 3.5 + 1.2) kg} = 0.0403 \text{ m}
\]

\[
\bar{y} = \frac{\sum_{i=1}^{3} m_i y_i}{\sum_{i=1}^{3} m_i} = \frac{m_1 y_1 + m_2 y_2 + m_3 y_3}{m_1 + m_2 + m_3} = \frac{(2.3485 + 0.6475 + 0.0252) kg \cdot m}{(5.5 + 3.5 + 1.2) kg} = 0.2962 \text{ m}
\]

Example 2: Table gives locations of hip, knee, ankle and 5th metatarsal at one instant. Suppose the center of mass of the thigh is 0.433 of the way from hip to knee, center of mass of leg is 0.433 from knee to ankle, and center of mass of foot is 0.50 of the way from ankle to 5th metatarsal head. Find the center of mass of each segment.

<table>
<thead>
<tr>
<th>Hip</th>
<th>Knee</th>
<th>Ankle</th>
<th>5th Met</th>
</tr>
</thead>
<tbody>
<tr>
<td>X</td>
<td>Y</td>
<td>X</td>
<td>Y</td>
</tr>
<tr>
<td>0.4474</td>
<td>0.7870</td>
<td>0.4075</td>
<td>0.4754</td>
</tr>
</tbody>
</table>

Use the following formulas, based on the description of the center of mass (CoM) locations in the problem:

- \( \text{ThighCoM}\_X = \text{Hip}_X + 0.433(\text{Knee}_X - \text{Hip}_X) \)
- \( \text{ThighCoM}\_Y = \text{Hip}_Y + 0.433(\text{Knee}_Y - \text{Hip}_Y) \)
- \( \text{ShankCoM}\_X = \text{Knee}_X + 0.433(\text{Ankle}_X - \text{Knee}_X) \)
- \( \text{ShankCoM}\_Y = \text{Knee}_Y + 0.433(\text{Ankle}_Y - \text{Knee}_Y) \)
- \( \text{FootCoM}\_X = \text{Ankle}_X + 0.50(\text{5Met}_X - \text{Ankle}_X) \)
- \( \text{FootCoM}\_Y = \text{Ankle}_Y + 0.50(\text{5Met}_Y - \text{Ankle}_Y) \)

<table>
<thead>
<tr>
<th>Thigh</th>
<th>Shank</th>
<th>Foot</th>
</tr>
</thead>
<tbody>
<tr>
<td>CoM X</td>
<td>CoM Y</td>
<td>CoM X</td>
</tr>
<tr>
<td>0.430</td>
<td>0.652</td>
<td>0.272</td>
</tr>
</tbody>
</table>

**Moment of Inertia: Theoretical Background**

Torque, moment of inertia, and angular acceleration are related by the equation

\[ \tau = I \alpha \]

where

- \( \tau \) = torque (N·m) (Some writers use M (for moment) for this quantity.)
- \( \alpha \) = angular acceleration (radian/s²)
- \( I \) = moment of inertia (kg·m²)

For masses distributed in three dimensions, the moment of inertia about the z axis is given by

\[ I_{z-axis} = \sum m_i (x_i^2 + y_i^2) \]

There are analogous formulas for moments about the x and y axes. If a segment is long and thin, and the long axis is oriented along x, then the \( y_i \)'s will be small compared to the \( x_i \)'s, and we can pretend that they
are zero without too much error. In that case, the equation for the moment of inertia about the z axis (i.e. about $x=0$) simplifies to

$$I_{z-axis} = \sum_i m_i x_i^2$$

The moment of inertia about the center of mass, located at $x=x_{cm}$, is found by replacing $x_i$ with $(x_i-x_{cm})$. We call the moment about the center of mass $I_0$.

$$I_0 = \sum_i m_i (x_i - x_{cm})^2$$

To be precise, we should specify the axis of rotation that passes through the c.m. If not otherwise specified, we assume that the axis of rotation is perpendicular to the long axis of the segment, and that the segment has sufficient symmetry that it doesn’t matter which “perpendicular-to-the-long-axis” axis we choose, since the moment of inertia will be about the same for any of them.

**Radius of Gyration**

Radius of gyration ($\rho_{gyr}$) is a measure of the “effective” distance of a solid object, or objects, from an axis, for “moment of inertia purposes”. More specifically, the radius of gyration is the distance (radius) from an axis to a point mass, such that a point mass at that distance will have the same moment of inertia as the original object or objects. In other words, the radius of gyration, $\rho_{gyr}$, satisfies the equation

$$I = m \rho_{gyr}^2$$

Radius of gyration depends on size and shape of an object but not on its mass.\(^1\) Radius of gyration is computed from moment of inertia (if the latter is known) as follows:

$$\rho_{gyr} = \sqrt{\frac{I}{m}}$$

The radius of gyration about the center of mass is denoted by adding a subscript 0: $\rho_0$. It is defined by the equation

$$I_0 = m \rho_0^2$$

This is a measure of how spread out the segment’s mass is, relative to its center of mass. $\rho_0$ is analogous to the standard deviation $\sigma$ in statistics, which measures how spread out the data is, relative to the mean. If we compare the equations for $I_0$ above, we see that

$$\sum_i m_i (x_i - x_{cm})^2 = m_{tot} \rho_0^2$$

$$\rho_0^2 = \frac{\sum_i m_i (x_i - x_{cm})^2}{m_{tot}}$$

---

\(^1\) If object has uniform density, the radius of gyration is independent of density and mass. If the object does not have uniform density then the radius of gyration depends on the “shape” of the density distribution, but it is unaffected by scaling all the densities in an object (such as doubling).
\[ \rho_0 = \frac{\sum_i m_i (x_i - x_{cm})^2}{m_{tot}} \]

which is very similar to the equation for the standard deviation

\[ \sigma = \sqrt{\frac{\sum_i n_i (x_i - \bar{x})^2}{N}} \]

where \( n_i \) is the number of observations when \( x=x_i \), and \( N \) is the total number of observations.

In general, if you replace a solid object with a point mass, you can position the point mass to get the center of mass “right”, or to get the moment of inertia right, but not both. If you want to get both right, you will (generally) need two point masses.

Example: A uniform thin rod of mass \( m \) and length \( L \) is centered at the origin. (a) Find the position of a point mass that has the same center of mass. (b) Find the position of a point mass that has the same moment of inertia about an axis through the origin. (c) Find the location of two point masses that have the same center of mass and the same moment of inertia as the rod.

a. Begin by computing the rod’s center of mass location and its moment of inertia about the origin. Its center of mass is determined by symmetry to be at \( \bar{x}_{rod} = 0 \). Since the center of mass is at the origin, we can use the formula for the moment of inertia of a rod about its center of mass to find the moment of inertia about the origin: \( I_{rod,origin} = mL^2/12 \). Point mass \( m \) at position \( x=0 \) has \( \bar{x}_{pm} = 0 \), which matches the rod center of mass location. Note that the moment of inertia about an axis through the origin is \( I_{pm,origin} = 0 \), since the distance of the mass from origin is zero. This does not match the rod’s moment of inertia about the origin.

b. Formula for moment of inertia of this thin rod about the origin is \( I_{rod,origin} = mL^2/12 \).

Therefore \( \rho_{gyr} = \sqrt{\frac{I}{m}} = 0.289L \). A point mass \( m \) at position \( x=\rho_{gyr}=0.289L \) will have moment of inertia \( I_{pm,origin} = m(0.289L)^2 = mL^2/12 \) which matches the rod’s moment of inertia about the origin. Note that the center of mass of this one-point-mass “system” is at \( \bar{x}_{pm} = 0.289L \), which does not match the center of mass of the rod.

c. Use two masses, each with half the total mass. Put one at \( x = +\rho_{gyr} \) and one at \( x = -\rho_{gyr} \). The center of mass will be at \( \bar{x}_{pm} = 0 \). The total moment of inertia about the origin is

\[ I_{pm} = I_1 + I_2 = \frac{m}{2} (0.289L)^2 + \frac{m}{2} (0.289L)^2 = mL^2/12 \]

The two-point-mass system has the same center of mass and the same moment of inertia about the origin as the rod.

The preceding example shows that one point mass in the right place can have the same center of mass or the same moment of inertia as a thin rod, but cannot reproduce both properties simultaneously. Two point masses \( (m/2) \) at the right places \((\bar{x} + \rho_{gyr} \) and \( \bar{x} - \rho_{gyr} \)) can reproduce both the center of mass and the moment of inertia for this one-dimensional system.
Moments of Inertia for Different Shapes

In the following formulas, \( m \) = mass and subscript \( 0 \) indicates that the axis of rotation passes through the center of mass.

Point mass:  
\[ I = m r^2 \]

Thin rod length \( L \), about perpendicular axis through center of mass:  
\[ I_0 = \frac{mL^2}{12} \]

Thin rod, about perpendicular axis through end:  
\[ I = \frac{mL^2}{3} \]

Solid cylinder:  
\[ I_{x0} = I_{y0} = \frac{m(3r^2 + h^2)}{12}, I_{z0} = \frac{mr^2}{2} \]

Frustum of cylinder, axis of symmetry \( z \), radii \( R \) and \( r \), height \( h \):  
\[ I_{z0} = \frac{3m(R^5 - r^5)}{10(R^3 - r^3)} \]

Volume:  
\[ \frac{\pi h}{3}(R^2 + Rr + r^2) \]

Center of mass, from \( R \) end:  
\[ \bar{z} = \frac{h}{4} \left( \frac{R^2 + 2Rr + 3r^2}{R^2 + Rr + r^2} \right) \]

(\( I_{z0} \) for frustum reduces to \( I_{z0} \) for cylinder when \( R = r \). Use l’Hopital’s rule to prove.)

Solid sphere:  
\[ I_0 = \frac{2mr^2}{5} \]

Solid ellipsoid with semi-major axes \( a, b, c \) along \( x, y, z \) respectively:  
\[ I_{x0} = \frac{m(b^2 + c^2)}{5}, I_{y0} = \frac{m(a^2 + c^2)}{5}, I_{z0} = \frac{m(a^2 + b^2)}{5} \]

Parallel Axis Theorem

Let \( I_1 \) = moment of inertia about axis 1, and let \( d \) = distance from axis 1 to parallel axis 2. Then  
\[ I_2 = I_1 + md^2 \]
Example: Parallel Axis Theorem

Question: What is the moment of inertia for a cylinder (mass m, length L, radius r) swinging about an axis perpendicular to the symmetry axis, through one end?

Answer: Use the formula for moment of inertia of a cylinder about a perpendicular axis through its center of mass. Compute distance of center of mass from end. Use parallel axis theorem.

\[ I_{x0} = \frac{m(3r^2 + L^2)}{12}, \quad d = \frac{L}{2} \]

\[ I_{x,\text{end}} = I_{x0} + md^2 = \frac{m(3r^2 + L^2)}{12} + \frac{ml^2}{4} \]

\[ I_{x,\text{end}} = \frac{m(3r^2 + 4L^2)}{12} \]

**Moment of Inertia of a System**

The moment of inertia of a system with multiple parts is the sum of the moments of inertia of the individual parts:

\[ I_{\text{total}} = \sum_{i=1}^{N} I_{i} \]

Therefore the moment of inertia of the arm about the shoulder is the sum of the moments of the upper arm about the shoulder, the forearm about the shoulder, and the hand about the shoulder. The moments if the individual segments can be computed by applying the parallel axis theorem.

**Estimating Segment Centers of Mass and Moments of Inertia**

Measure subject height H and mass m. If possible, measure segment lengths using the same landmarks as in the tables to be used.

Use Winter’s Table 4.1 to estimate segment mass from body mass. Use measured segment length if available, otherwise estimate segment length from Figure 4.1. If you have detailed anatomical data (for example, scans or circumference measurements), you may be able to estimate \( I_0 \) for a segment using the equation for \( I_0 \) given above and reproduced here:

\[ I_0 = \sum_i m_i (x_i - x_{cm})^2 \]

If, however, you do not have detailed anatomical data, you can still estimate \( I_0 \) using the equation

\[ I_0 = m_{\text{tot}}\rho_0^2 \]

where \( m_{\text{tot}} \) is the segment mass and \( \rho_0 \) is the radius of gyration of the segment about its center of mass.

Table 4.1 in Winter 4th ed., 2009, specifies the radius of gyration, \( \rho_0 \), of various segments about their centers of mass (which he calls the center of gravity, or C of G), as a fraction of the segment length.

Use Table 4.1 to estimate distance to segment center of mass and segment radius of gyration, based on measured or estimated segment length. Compute segment moment of inertia using the formula

\[ I_{\text{seg}} = m_{\text{seg}}\rho_{\text{gyr}}^2 \]

Winter’s Table 4.1 gives moments about axes perpendicular to the long axis of the segment. It does not give formulas for moments about the long axis. Winter’s Table 4.1 lists three radii of gyration for most
segments: about the center of mass and about the proximal and distal ends. The latter two radii of gyration are useful “shortcuts” for problems in which the rotation is about the proximal or distal end. The user does not have to apply the parallel axis theorem to the moment about the center of mass, because Winter has already done it for us. For example, one uses the estimated segment mass and the radius of gyration about the proximal joint to compute the moment of inertia for rotation about the proximal joint: $I_{seg.prox} = m_{seg} \rho_{gyr.prox}^2$.

Figures and a table from Winter, 4th ed., 2009, follow.

Figure 4.1  Body segment lengths expressed as a fraction of body height $H$. 
Figure 4.2  Density of limb segments as a function of average body density.
\[
I_{\text{prox}} = I_0 + m r_{\text{prox}}^2 = m r_{\text{prox}}^2 + m r_{\text{prox}}^2
\]

\[
\Rightarrow m r_{\text{prox}}^2 = m (r_{\text{prox}}^2 + r_{\text{prox}}^2) = r_{\text{prox}}^2 + r_{\text{prox}}^2
\]

<table>
<thead>
<tr>
<th>Segment</th>
<th>Definition</th>
<th>Segment Weight/Total</th>
<th>Center of Mass/Segment Length</th>
<th>Radius of Gyration/Segment Length</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Body Weight Proximal</td>
<td>Distal C of G Proximal Distal</td>
<td></td>
</tr>
<tr>
<td>Hand</td>
<td>Wrist axis/knuckle II middle finger</td>
<td>0.006 M</td>
<td>0.506 0.787 0.577 M P</td>
<td>0.16</td>
</tr>
<tr>
<td>Forearm</td>
<td>Elbow axis/ulnar styloid</td>
<td>0.016 M</td>
<td>0.430 0.507 0.647 M P</td>
<td>1.13</td>
</tr>
<tr>
<td>Upper arm</td>
<td>Glenohumeral axis/elbow axis</td>
<td>0.028 M</td>
<td>0.436 0.564 0.645 M P</td>
<td>1.07</td>
</tr>
<tr>
<td>Forearm and hand</td>
<td>Elbow axis/ulnar styloid</td>
<td>0.022 M</td>
<td>0.682 0.318 0.656 P</td>
<td>1.14</td>
</tr>
<tr>
<td>Total arm</td>
<td>Glenohumeral joint/ulnar styloid</td>
<td>0.050 M</td>
<td>0.530 0.470 0.596 P</td>
<td>1.11</td>
</tr>
<tr>
<td>Foot</td>
<td>Lateral malleolus/head metatarsal II</td>
<td>0.0145 M</td>
<td>0.50 0.50 P 0.690 0.690 P</td>
<td>1.10</td>
</tr>
<tr>
<td>Leg</td>
<td>Femoral condyles/medial malleolus</td>
<td>0.0465 M</td>
<td>0.433 0.567 0.643 M P</td>
<td>1.09</td>
</tr>
<tr>
<td>Thigh</td>
<td>Greater trochanter/femoral condyles</td>
<td>0.100 M</td>
<td>0.50 0.50 P 0.653 M P</td>
<td>1.05</td>
</tr>
<tr>
<td>Foot and leg</td>
<td>Femoral condyles/medial malleolus</td>
<td>0.061 M</td>
<td>0.476 0.554 0.650 P</td>
<td>1.09</td>
</tr>
<tr>
<td>Total leg</td>
<td>Greater trochanter/medial malleolus</td>
<td>0.161 M</td>
<td>0.447 0.553 0.650 P</td>
<td>1.06</td>
</tr>
<tr>
<td>Head and neck</td>
<td>C7–T1 and 1st rib/ear canal</td>
<td>0.081 M</td>
<td>1.000 — 0.495 0.116 — PC</td>
<td>1.11</td>
</tr>
<tr>
<td>Shoulder mass</td>
<td>Sternoclavicular joint/glenohumeral axis</td>
<td>—</td>
<td>0.712 0.288 —</td>
<td>1.04</td>
</tr>
<tr>
<td>Thorax</td>
<td>C7–T1/T12–L1 and diaphragm*</td>
<td>0.216 PC</td>
<td>0.82 0.73 0.92</td>
<td>0.92</td>
</tr>
<tr>
<td>Abdomen</td>
<td>T12–L1/L4–L5*</td>
<td>0.139 LC</td>
<td>0.44 0.56</td>
<td>—</td>
</tr>
<tr>
<td>Pelvis</td>
<td>L4–L5/greater trochanter*</td>
<td>0.142 LC</td>
<td>0.105 0.895</td>
<td>—</td>
</tr>
<tr>
<td>Thorax and abdomen</td>
<td>C7–T1/L4–L5*</td>
<td>0.355 LC</td>
<td>0.105 0.895</td>
<td>—</td>
</tr>
<tr>
<td>Abdomen and pelvis</td>
<td>T12–L1/greater trochanter*</td>
<td>0.281 PC</td>
<td>0.105 0.895</td>
<td>—</td>
</tr>
<tr>
<td>Trunk</td>
<td>Greater trochanter/glenohumeral joint*</td>
<td>0.497 M</td>
<td>0.50 0.50</td>
<td>1.03</td>
</tr>
<tr>
<td>Trunk head neck</td>
<td>Greater trochanter/glenohumeral joint*</td>
<td>0.578 MC</td>
<td>0.66 0.34 P 0.830 0.607 M</td>
<td>—</td>
</tr>
<tr>
<td>Head, arms, and trunk (HAT)</td>
<td>Greater trochanter/glenohumeral joint*</td>
<td>0.678 MC</td>
<td>0.626 0.374 PC 0.798 0.621 PC</td>
<td>—</td>
</tr>
</tbody>
</table>

1 NOTE: These segments are presented relative to the length between the greater trochanter and the glenohumeral joint.