The Algebraization of Fraction Division: A Unit to Support Lasting Understanding of Concepts and Algorithms in Sixth Grade Mathematics

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Introduction

Across the grades, two mathematical topics in particular are consistently identified as tripping points and "gatekeepers" for students: in the elementary and middle grades, this topic is fractions[1]; in the later high school years, this topic is algebra.[2] Importantly, these conceptual fields do not exist in isolation. Studies have pointed out that success or struggle with fractions is strongly associated with success or struggle in algebra.[3]

Though separate topics, further investigation shows that the relationship between fraction mastery and algebraic fluency is not arbitrary. Both concepts heavily depend upon relational thinking, or recognizing consistency across number, operations and equality despite the context.[4] When arithmetic with fractions is taught in ways that value the structure of the operation being performed, students are more likely to become adept in the generalization processes necessary for algebraic thought.[5] This process of leveraging students' thinking in arithmetic to promote the styles of thinking necessary for algebra is known as *algebraization*.[6]

Importantly, sixth grade is the last time that fractions are formally addressed as a content standard under the Common Core, focusing attention on fraction division.[7] Sixth grade becomes a critical moment, as the mathematical differences between students' fraction achievement at this age becomes amplified by eighth grade.[8] For these reasons, effectively teaching the concept of fraction division in sixth grade is of critical importance not only within the study of fractions themselves, but also in preparation for algebra and other mathematics to follow.

In teaching the topic, then, "mastery" of fraction division must blend both conceptual and procedural goals. It is not sufficient for a student to simply carry an algorithm with them to later grades, nor is it sufficient for a student to build and not formalize conceptual work with fractions into efficient schemes for solving problems. Indeed, fraction concepts and procedures are said to develop concurrently, reinforcing one another.[9] Concepts and procedures must co-exist as priorities of instruction for both teacher and students.

Taken together, effective teaching of fraction division hinges upon authentic connection: between arithmetic and algebraic reasoning, concepts and procedures, and context and abstraction. This unit focuses on supplementing pre-existing curricular materials that already place heavy emphasis on conceptual foundations in order to reinforce procedural understandings and connections to other areas of mathematics.

Rationale

This unit has been motivated by my teaching context and the material of the seminar I have attended during this cycle, called Computational Thinking. I teach sixth and eighth grade mathematics at John Dickinson Middle Years Programme in Wilmington, Delaware, though the unit that follows will focus exclusively on sixth grade content of fraction division. As a magnet school, students enter sixth grade mathematics with a variety of elementary school experiences either within our own district (Red Clay Consolidated School District) or from other districts nearby. Students must apply to enter the program, but academic achievement is not a criteria. Since our program does not put students into separate ability tracks but culminates in an algebra course by eighth grade, I am seeking to develop a unit that is accessible to and meaningful for students of a wide range of ability levels within the same classroom.

My school is also nearing the final stages of its application as an official International Baccalaureate Middle Years Programme. The International Baccalaureate program values high-level mathematical thinking as well as an interdisciplinary approach to learning, motivating the focus on context and application that occurs later in the description of the unit. Additionally, the program values the development of approaches to learning skills and character traits that support students to become responsible global citizens. These influences have also shaped the objectives for the unit that follows.

Mathematical Content: Fraction Division

In supporting students to master fraction division to high levels, there are several components that must be considered. First, often the ability to divide fractions is limited by interpretations of fractions themselves, as they exist as numbers, operations and relationships simultaneously.[10] Secondly, division as an operation requires flexibility and a strong understanding of its interpretation in relation to multiplication. Finally, it is important to know that multiple procedural algorithms exist for dividing fractions, and flexibility in application of algorithms should depend on the problem presented. The following section reviews the mathematical content behind each of these potential tripping points for students that are necessary for true conceptual and procedural fluency with the topic.

What is a Fraction?

According to Behr and Post[11], there are at least five ways that fractional quantities may be interpreted:

1. As a *part-whole* comparison, through which we have a number of parts (the numerator) per number of equal-sized parts (the denominator).
2. As an *operator*, in the instances where one might take a fraction "of" another quantity.
3. As a *quotient*, through which a fraction is interpreted as a division of quantities.
4. As a *ratio*, through which a fraction is interpreted as a comparison of quantities.
5. As a *measure*, or a quantity that can be plotted on a number line in relation to other rational quantities.

In addition to these perspectives, other research includes yet another category of fractional interpretation: a nuance to the "part-whole" comparison known as an *iterative* interpretation.[12] According to the iterative perspective, a fraction such as m/n is interpreted not as "m out of n parts", but rather m iterations of 1/n parts. This distinction makes work with fractions larger than one even more meaningful, and also happens to be a key difference in how Japanese students interpret fractions over U.S. students.[13]

What is Division?

Division is a statement that inverts a multiplicative relationship. Division statements of the form $a \div b = c$ can be interpreted in one of two ways:

- According to the quotative division model, I have *a*, and wish to place *b* in each group. How many groups can I make? The answer is *c*.
- According to the partitive division model, I have *a*, and wish to make *b* groups. How much is in each group? The answer is *c*.[14]

Knowing that there are these two ways to interpret divisions are critical for making sense of division statements, as often with fractions, one interpretation can be more useful than others. For instance, in the case of $\frac{3}{4} \div \frac{1}{8}$, we could interpret as:

I have $\frac{3}{4}$, and wish to place $\frac{1}{8}$ in each group. How many groups can I make?(see Figure 1)

1	2	3	4	5	6		

Figure 1

From the diagram above where $\frac{3}{4}$ are shaded in grey, we see that 6 groups of $\frac{1}{8}$ can be created, so $c = 6$. Alternatively, we could interpret the same problem as:

I have $\frac{3}{4}$, and wish to make it $\frac{1}{8}$ of a group. How much is in a group?

Under this interpretation, if $\frac{3}{4}$ is $\frac{1}{8}$ of a group, then 8 groups of $\frac{3}{4}$ will make one full group. We can see from the visual below that 8 groups of $\frac{3}{4}$ should give us 6 wholes, so again $c = 6$. Figure 2 shows this.

Group 1	Group 1	Group 1	Group 7

Group 2	Group 2	Group 2	Group 7

Group 3	Group 3	Group 3	Group 7

Group 4	Group 4	Group 4	Group 8

Group 5	Group 5	Group 5	Group 8

Group 6	Group 6	Group 6	Group 8

Figure 2

Both interpretations arrive at the same result because both are valid reversals of a multiplicative process. $b \times c = a$ could mean that b groups of c result in a or that c groups of b result in a. Thus, division may be the inverse of either of these two interpretations. Students must be flexible with these interpretations as context and style of problem may lend themselves more readily to one perspective over another.

What Algorithms Exist for Dividing Fractions?

In reality, there are likely an infinite number of small variations of fraction division algorithms that could make sense to apply in a given situation. Just as particular interpretations of division are more helpful in certain problems than others, similarly different algorithms may be better fits for some problems over others, as well.

The Standard Algorithm

The standard algorithm for fraction division relies upon division's definition as the inverse of a multiplicative relationship. It reasons that dividing by a quantity is the same as multiplying by the multiplicative inverse, or reciprocal, of a quantity. For instance, a division of the form $\frac{a}{b} \div \frac{c}{d}$ could be re-written as $\frac{a}{b} \times \frac{d}{c} = \frac{ad}{bc}$.

Unfortunately, all too often, this algorithm is the only one learned and retained by students. Videos that further turn this algorithm into a "magic trick" to divide fractions, dubbed "Keep Change Flip" or "KFC" for short, can be found in mass on Youtube.[15] Reliance exclusively on this algorithm also attributes to some of the more common procedural errors with fraction division, such as "flipping" the numerator and denominator of the dividend as opposed to the divisor [16].

Regardless of the issues it may cause if incorrectly generalized or conceptually decontextualized, teaching of this algorithm is still crucial to students' later development in algebra. Indeed, if students are working with divisions of rational functions or simply solving equations with rational values, knowledge of a reciprocal and its relationship to division are necessary. However, it is also important for students to know that this algorithm is not always the most efficient choice when solving a problem.

Common Denominator Algorithm

There is some support for delaying the introduction of the standard algorithm for division in favor of emphasis of the common denominator algorithm. Through the common denominator algorithm, a division problem such as $\frac{3}{4} \div \frac{1}{8}$ could be reimagined as $\frac{6}{8} \div \frac{1}{8}$. The question of "how many groups of $\frac{1}{8}$ fit into $\frac{6}{8}$ can then be answered through a division of numerators, as $6 \div 1 = 6$.

Emphasis of this algorithm presents its strengths in making multiplicative comparisons and the operation of division itself explicit to students [17]. Indeed, knowledge of this interpretation could also support the process of algebraization when students extend concepts of division to work with "like terms" (i.e., in the recognition that $6a \div 1a$ would most certainly be 6). In problems with very unlike denominators, however, this procedure could prove more challenging than the standard algorithm to apply.

Additional Algorithms

Though these two algorithms are the most commonly discussed, other algorithms for dividing fractions do exist. A student, for instance, once reasoned that, since ratio relationships can be represented as divisions, a division of fractions could be thought of as the writing of a proportional relationship. For instance, if $\frac{\frac{3}{4}}{\frac{1}{8}}$, then this ratio could be scaled by a factor of 8 to result in an equivalent ratio of $\frac{6}{1}$. Indeed, when conceptual connections are clear to students, there is no limit to the procedural nuance they might be able to generalize and apply in constructing an algorithm.

Objectives

Below, unit objectives have been organized according to mathematical goals for the content taught and the practices students will be expected to develop.

Mathematical content objective: Division of fractions

This unit will be developed to supplement curricular materials from Illustrative Mathematics to support the Common Core Standard 6.NS.1:

> *Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem.*

Illustrative Mathematics, a free curricular resource that our school district has adopted this year, provides a clear map of how students should move from conceptual to procedural fluency with fraction division through their lesson structure, shown in Figure 3.

Target	Summary of Emphasis	Lessons
Making Sense of Division	Introduces students to strategies of determining reasonability of a quotient and interpreting division as the inverse of a multiplication.	3
Meanings of Fractions (Partitive Model)	Supports students to develop the interpretation of division as answering the question of “how many groups”. Emphasizes models for sense-making and also pushes students toward reasoning about partial groups.	4
Meanings of Fractions (Quotitive Model)	Supports students to develop the interpretation of division as answering the question of “how much in each group”.	2
Algorithm for Fraction Division	Supports students to develop the standard algorithm of fraction division, through reasoning that dividing by a/b is the same as multiplying by b and then multiplying by 1/a.	2
Fractions in Lengths, Areas, and Volumes	Applies division to answer questions of how many times as much in the context of problems of missing lengths, areas of rectangles and triangles, and volumes.	4
Let’s Put It Together	Applies division to answer a wide variety of problems using either interpretation or algorithm.	1

Figure 3

As is shown in the table above, the curriculum spends significant time articulating the dual interpretations of division and conceptual reasoning involved and does so in ways that are very meaningful. However, less time is given to the process of developing an algorithm (or even algorithms) that students may use to solve future problems. It is here that I intend to construct supplemental activities that 1) generalize algorithms from structured study of patterns in divisions of fractions, 2) communicate clear connections between algorithms, visuals, and division statements, and 3) continue to present problems in which the resources developed may be applied to solve problems relevant to not only study of geometry but also to other problems in the real-world. The goal here is to ensure that students solidify procedures that are still rooted in the conceptual foundations built throughout the unit.

Mathematical practice objectives: Criteria B, C, and D

The International Baccalaureate program also outlines additional goals for mathematics instruction, made explicit to teachers and students through what are known as assessment criteria. These expected outcomes are consistent throughout grades 6 through 10, but the level of abstraction and complexity at each grade level steadily increases as students mature. In sixth grade, the following assessment criteria are applied:

1. Knowing and understanding of the mathematics, culminating in the ability to solve challenging and unfamiliar problems using tools developed throughout the course (Criterion A).
2. Investigating patterns in mathematics, culminating in the ability to apply mathematical problem-solving techniques to discover complex patterns, describe patterns as relationships and/or general rules consistent with findings, and verify whether the patterns works for other examples (Criterion B).
3. Communicating mathematical ideas, culminating in the ability to use appropriate mathematical language and representations in both oral and written statements, communicate complete and coherent mathematical lines of reasoning, and organize information using a logical structure (Criterion C).
4. Applying mathematics in the real-world, culminating in the ability to identify relevant elements of authentic real-life situations, select appropriate mathematical strategies and apply them successfully to reach a solution, and explain the degree of accuracy of a solution and whether it makes sense in the context of the authentic real-life situation (Criterion D).

Fraction division provides opportunities for all of these assessment criteria to be highlighted and emphasized, which will not only contribute to meeting the goals of the International Baccalaureate program but will also further deepen students’ understanding and fluency as they are supported to develop these mathematical learning practices.

Teaching Strategies

To achieve the goals of the unit, a computational thinking framework may be applied as the primary teaching strategy. Computational thinking has been defined in two distinct ways that I intend to leverage during this unit: 1) as instructional, geared toward supporting abstraction of a class of mathematical ideas 2) as dispositional, shifting students' relationships to mathematics and its connections to our world.

The instructional lens on computational thinking emphasizes breaking apart problems, discovering commonalities, and generalizing solutions such that they "can be effectively carried out by an information-processing agent".[18] In studying and emphasizing structural components and commonalities of fraction division, I hope to emphasize the importance of algorithms without leaning away from a conceptual bent. Since this unit involves a lot of re-emphasis and adaptation of already very conceptual materials, the challenge will be to weave this perspective throughout in ways that enhance the current materials.

The dispositional lens on computational thinking promotes complex, open-ended problems as a starting point to develop learning perspectives, such as tolerance for ambiguity, persistence through difficult problems, and collaboration with others to achieve a common goal, essentially generalizing mathematical thinking itself beyond the mathematics classroom.[19] It is through connections such as these that I can ensure consistency in a conceptual message and also integrate new purposefulness to the development of an algorithm for fraction division. Again, activities for this unit must include adaptations of current materials so that they integrate even more sustained and meaningful activity around fraction division.

There are two concrete ways that computational thinking dispositions can be even more fully integrated into the unit: 1) through integration of problem posing in addition to problem solving and 2) through the establishment of an authentic global context. Problem posing, directly aligned with the process of scientific inquiry, involves student generation of both questions and solutions, allowing for a process of exploration and deep mathematical thinking to meet instructional goals.[20] An authentic global context could also provide a landscape for this exploratory process with division of fractions to take place, consistently emphasizing meaningful connections between mathematical concepts and its value in our world. Indeed, research has demonstrated that authentic contexts of interest can make a difference in students' engagement and performance in mathematical problem-solving.[21]

Inherent to fraction division is the idea of comparison of quantities, and quantities have the power to reflect important attributes of our community and world. Therefore, in alignment with the International Baccalaureate's global context of identities and relationships, the statement of inquiry for this unit will center around the following exploration: The patterns in fraction division can be generalized to illustrate relationships between quantities that describe our local and global communities. This central theme and focus will be applied in both the supplemental activities and modification of the unit, motivating concrete teaching strategies such as explicit connection and collaboration.

Finally, in addition to these strategies, routines and supports now common to mathematics classrooms must also be considered in the implementation of the activities to follow. For instance, student familiarity with Number Talk routines is a critical component in the promotion of mathematical reasoning to occur. Number Talk structures are brief, non-evaluative conversations that engage a class in sense-making around a single computational problem.[22] Through engagement in Number Talks, students are exposed to a variety of strategies to solve a single problem, tasked with deep listening about new ideas, respecting other ways of thinking, and ultimately learning from and with others in the classroom. As students work toward the generalization of a procedure for fraction division, Number Talks become a powerful tool.

Since collaboration, conversation and ultimately connection of ideas are also needed to reach the twin goals of conceptual and procedural fluency with fraction division, supports for discussion among students are also critical. To encourage structure in student-to-student collaborations, supports such as sentence starters will be employed. Sentence starters not only have the power to assist students in fine-tuning their use of academic language, but can also aid students in making explicit the need for justification and reasoning behind the ideas they share.[23]

To facilitate high-level analysis of student ideas and connections in full-class discussions, the five steps from "5 Practices for Orchestrating Productive Mathematical Discussions" will be used regularly in each activity. These five steps—anticipating, monitoring, selecting, sequencing, and connecting—describe the core activities in which a teacher must engage to ensure that discussions of student ideas productively arrive at the learning goal and highlight student ideas in that process.[24] The first step, anticipation of student solutions to problems, has been addressed briefly within each activity that utilizes full class discussion techniques. From there, the teacher will need to closely monitor student work as it occurs in the classroom, select strategies and solutions to share during discussion moments, sequence appropriately to reach the desired learning outcome of each activity, and finally ensure that supports are in place for students to make the desired connections within each task.

Activities

Activity 1: The Global Community Project

The following activity is intended to serve as an anchor throughout the unit. The questions asked of students and the ways that they are thinking about the tasks are expected to change as they develop new understandings of the meanings of division and fractions.

After the first few lessons on the meaning of division, the global context can be launched with students, centered around the following central question: How do our day-to-day communities compare to the global community? They will work throughout the unit with a common dataset from the concept of "If the World Was Only 100 People".

In the first phase of the unit, students begin by thinking about their own communities. Some discussion should occur around the many communities that we are part of so that students can recognize that they are members of many different communities in our local area. Discussion should also occur around what a sample of these communities would mean or look like. Next, students will be tasked with identifying a sample of exactly 10 people with whom they interact regularly. They should survey these people with any three of the following questions to compare to the 100 people dataset:

- How old are you?
- What continent are you from?
- What is your religion?
- What is your first language?
- Do you have a cell phone?
- Do you actively use the Internet?

Once students have collected their information, they will be challenged to compare their datasets with the 100 people set, presenting their results in the form of a small booklet. An invented dataset could be used as a starting point to show examples of what this might look like. In drawing comparisons, we want to emphasize that our question will be, "how many times more/less of this characteristic exists in my community versus in the larger global community?" Expectations for the final booklet product should include support with visuals and/or calculations for every finding they describe.

Attention should be brought to the fact that, while our sample was only of 10 people, the statistics we are working with globally are for 100 people. Though fraction division is the focus of this unit, it is critical that this activity remain authentic by allowing students to select the strategy of their choice in solving this problem. Ratios could also be very easily applied to draw conclusions, for instance. The important idea here is that students can connect all of these approaches as relating to the same central concepts of proportion and division, allowing them to broaden their understanding of fraction division as a whole.

To emphasize the relevance that fraction division problems have in describing authentic contexts, at least one conversation prior to the final submission of the booklet should be dedicated to open problem-posing. Students should be given the following instruction at the launch of a lesson that has not yet been explicitly connected back to this project. Students can be told: “Please pose as many mathematical problems as you can that could be solved with the equation $\frac{75}{100} \div \frac{1}{4}$.” In the open discussion that follows, connections could be made back to the project context in ways that could assist them in interpreting division in their final products.

As students work, students should be made aware that the activity will culminate in a final reflection about their connections to the global context and to the mathematics of the unit. These questions could be posted on a board or included in a project description as students work:

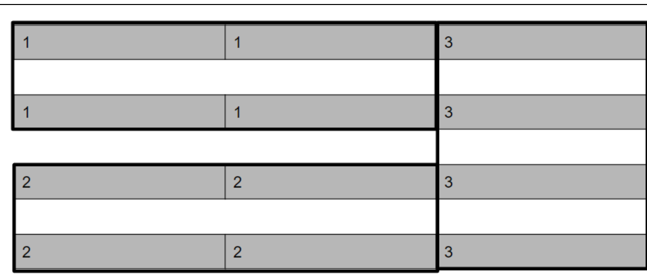
- How are fractions involved in your comparisons?
- How can division help us to draw comparisons?
- How can you summarize what you have learned about how your community compares to the larger global community?

When students have finished their booklets comparing the data, the multiple strategies that students create should be selected, sequenced and presented to the class to conclude the discussion. A gallery walk format (either physically in the room or virtually for even deeper analysis) might be helpful to achieve this goal. It is essential that, during this presentation, strategies that involve visuals, fractions, and explicit divisions are brought to the forefront. To conclude the activity, students should author their final reflections using their own as well as these new ideas.

Activity 2: Fraction Division Card Sort

The following two activities are designed to replace the “Algorithm for Fraction Division” section of the Illustrative Mathematics materials, extending time spent on this topic slightly and emphasizing culmination of the conceptual development of fraction division in a concrete algorithm.

In the first activity of this lesson, students will be presented with a set of several fraction division problems, visuals, and solutions (shown in Figure 4). Their task will be to work collaboratively with a partner to find the matches across these cards. Cards that do not have a match may require that students write their own solutions.

$\frac{4}{1} \div \frac{4}{3}$		3

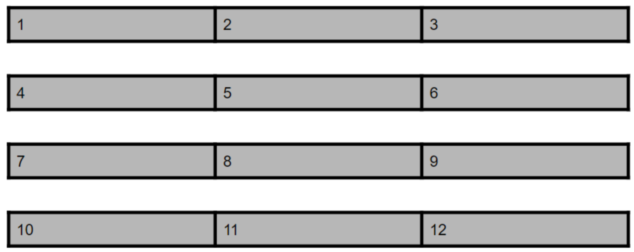
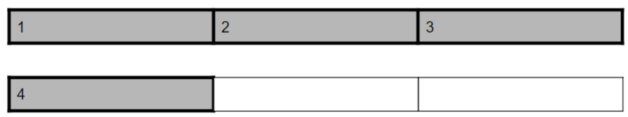
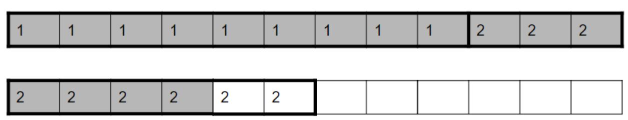
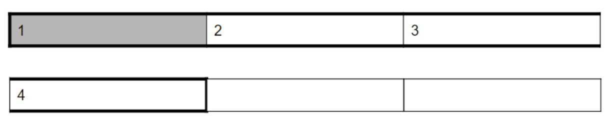
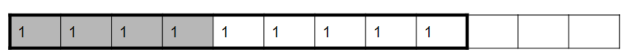
$\frac{4}{1} \div \frac{1}{3}$		12
$\frac{4}{3} \div \frac{1}{3}$		4
$\frac{4}{3} \div \frac{3}{4}$		16/9
$\frac{1}{3} \div \frac{4}{3}$		1/4
$\frac{1}{3} \div \frac{3}{4}$		4/9

Figure 4

It is important when completing card sort activities to set expectations around what work with a partner should look like so that work is authentic and meaningful for both partners. A list like the following will be presented to students before beginning the activity:

- Each partner should take a turn making a match.
- When it is your turn to match, you should say something like, “I think that these cards match because…”
- If your partner has made a match, decide if you agree or disagree. Then, you should respond with something like, “I agree/disagree with this match because…”
- Be sure to listen carefully to your partner’s perspective. Ask questions to understand if you are not sure about their ideas. Only make matches when you both agree.

- Both partners should be able to explain and justify every match made by the end of the activity.

Active monitoring of student strategies during this time is critical in gauging student success with this activity as well as to identify the variety of strategies and structures that students are observing to make their matches. The teacher should be circulating consistently during the activity, taking time to listen carefully to both partners as they justify their matches. Questions to ask may include:

- "Why would this division expression match this visual?"
- "How can we see the solution to this division in the visual?"
- "Which matches in the set have been easiest to make? The most difficult? Why?"
- "Are there any matches you have made that have helped you to make other matches? Why?"

To culminate the work on the first day of this activity, students will be asked to prepare for an individual video recording of their work and thinking. Each partner takes a turn to select one of the matches they made until they have chosen three matches total. The video they create should be no longer than a minute and a half to explain the matches that they made and how they knew these would be matches. Video has been selected to show this work because of its affordances in students' ability to both share visual representations and verbal explanation of their choices.

The ideas from the card sort will re-emerge with extension in the following activity, taking correct student matches as a given. Depending on student performance in this task and time available, the teacher may wish to close the conversation during the close of the lesson or the launch of the following day's lesson. To do this, it is recommended that the teacher select a subset of matches from a partnership in the room or invent matches that reflect most common lines of thinking. These matches should include a mixture of correct and incorrect responses, and the class should be tasked with providing feedback to this group. Through the discussion, students should be encouraged to listen to one another and revise their own thinking.

Activity 3: An Algorithm for Fraction Division

Cards from the card sort were designed intentionally to have parallels that fit into one of two categories: 1) rational numbers divided by unit fractions and 2) rational numbers divided by non-unit fractions. In this activity, students will search for patterns in the problem and the solution of similarly structured problems to articulate an algorithm (most likely the standard algorithm) for fraction division.

To launch the conversation central to the day's desired outcome, the teacher should be prepared to talk about the concept of an algorithm as repeatable steps to reach a desired result. It is particularly important to connect this understanding back to the mathematics, as this is the main goal of the task. One way to do this might be through launching with a Number String involving the multiplication of single-digit by two-digit numbers, such as the following:

- 8×25
- 9×32
- 6×73

As the teacher records strategies for student work, it is critical that all solutions are left on the board. It is recommended that a teacher use a single color to record strategies in this case as color will be used later to identify commonalities across strategies.

After providing instruction about what an algorithm is, the teacher can task the class:

- What strategies show a common algorithm that we can apply to multiply a single-digit number by a two-digit number? (Note: color-coding similar strategies could help with this step)
- How could we describe this algorithm as a series of steps for any single-digit and two-digit number? Why do we need to perform this step?
- Will this algorithm work for any single-digit number multiplied by any two-digit number? How do you know?

Once the concept of an algorithm has been defined with this mathematical example, students will start with new problems that include cards from the previous activity organized in table format. This design is intentional so as to facilitate student connections between problems that are similar to one another. A sample of this format is shown in Figure 5, though the exact version provided to students should include more space in the boxes to write out work and connections.

		DIVIDEND					
DIVISOR		$\frac{1}{3}$	$\frac{3}{4}$	$\frac{4}{3}$	$\frac{4}{1}$		
	$\frac{1}{3}$	$\frac{1}{3} \div \frac{1}{3} =$	$\frac{3}{4} \div \frac{1}{3} =$	$\frac{4}{3} \div \frac{1}{3} =$	$\frac{4}{1} \div \frac{1}{3} =$	$\div \frac{1}{3} =$	$\div \frac{1}{3} =$
	$\frac{3}{4}$	$\frac{1}{3} \div \frac{3}{4} =$	$\frac{3}{4} \div \frac{3}{4} =$	$\frac{4}{3} \div \frac{3}{4} =$	$\frac{4}{1} \div \frac{3}{4} =$	$\div \frac{3}{4} =$	$\div \frac{3}{4} =$
	$\frac{4}{3}$	$\frac{1}{3} \div \frac{4}{3} =$	$\frac{3}{4} \div \frac{4}{3} =$	$\frac{4}{3} \div \frac{4}{3} =$	$\frac{4}{1} \div \frac{4}{3} =$	$\div \frac{4}{3} =$	$\div \frac{4}{3} =$
	$\frac{4}{1}$	$\frac{1}{3} \div \frac{4}{1} =$	$\frac{3}{4} \div \frac{4}{1} =$	$\frac{4}{3} \div \frac{4}{1} =$	$\frac{4}{1} \div \frac{4}{1} =$	$\div \frac{4}{1} =$	$\div \frac{4}{1} =$

		$\frac{1}{3} \div$	$\frac{3}{4} \div$	$\frac{4}{3} \div$	$\frac{4}{1} \div$	$\div \quad =$	$\div \quad =$
		$\frac{1}{3} \div$	$\frac{3}{4} \div$	$\frac{4}{3} \div$	$\frac{4}{1} \div$	$\div \quad =$	$\div \quad =$

Figure 5

Students will be assigned to complete this worksheet in groups of four, tasked with the following assignment:

- Fill in the solutions to the division problems from the previous lesson's card sort first. Then, answer the remaining division problems using any strategy you choose.
- What patterns do you notice as you work that help you to solve each problem? Take notes in the margins or space at the bottom of the page.
- Choose any other two fractions to fill in the blank spaces in the top row and first column. Try to apply the patterns you noticed in the previous problems to complete these new problems.

At this stage, it is crucial that connections are made explicit through full class conversation at various stopping points throughout the activity. Observations about patterns should be kept in a list visible to students as they continue to work and explore. If the connection to the standard algorithm does not come out in discussion, it is important that the teacher suggest this pattern to the class for evaluation.

Finally, students will work independently to author a report about an algorithm that can be applied to divide any rational number by a non-unit fraction. This task should be assigned to students independently, as they apply their ideas from group work to draw their conclusions. The task should prompt students to do the following:

- Write an algorithm to divide any number by a non-unit fraction. For each step in your algorithm, explain why we need to perform this step.
- How did you invent this algorithm? Discuss some of the discoveries, patterns or inspirations that helped you to create this list of steps.
- Show that your algorithm will work for at least two examples. For each example, be sure to show how you have applied each step in your algorithm and why this step would be required using a visual. You may use work from your invented two fractions for this section if you choose.

This report is certainly a culminating activity that students should take time to complete. It is recommended that students complete this report through a combination of in-class and homework time, submitting a final product that could be graded according to Criterion B as a summative assessment.

Conclusion

The solidification of fraction division concepts and algorithms are crucial for later mathematical success. The generalization process or algebraization of fraction division will support the development of structures that students will need to succeed in later algebra. The activities designed in this unit reinforce present curricular materials to emphasize these dual goals and provide a final connection to analysis of an authentic global context. Taken together, students will leave the unit with the foundations, dispositions, and skills necessary for lasting success in this area and other topics of mathematical learning.

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Appendix A: Common Core Standards

The following Common Core Standards for grade 6 are addressed in this unit:

CCSS.MATH.CONTENT.6.NS.A.1

Interpret and compute quotients of fractions, and solve word problems involving division of fractions by fractions, e.g., by using visual fraction models and equations to represent the problem. For example, create a story context for (2/3) ÷ (3/4) and use a visual fraction model to show the quotient; use the relationship between multiplication and division to explain that (2/3) ÷ (3/4) = 8/9 because 3/4 of 8/9 is 2/3. (In general, (a/b) ÷ (c/d) = ad/bc.) How much chocolate will each person get if 3 people share 1/2 lb of chocolate equally? How many 3/4-cup servings are in 2/3 of a cup of yogurt? How wide is a rectangular strip of land with length 3/4 mi and area 1/2 square mi?.

CCSS.MATH.CONTENT.6.NS.B.2
Fluently divide multi-digit numbers using the standard algorithm.

CCSS.MATH.CONTENT.6.RP.A.3
Use ratio and rate reasoning to solve real-world and mathematical problems, e.g., by reasoning about tables of equivalent ratios, tape diagrams, double number line diagrams, or equations.

Additionally, the following standards for mathematical practice are directly utilized in this unit:

CCSS.MATH.PRACTICE.MP1 Make sense of problems and persevere in solving them.

CCSS.MATH.PRACTICE.MP2 Reason abstractly and quantitatively.

CCSS.MATH.PRACTICE.MP7 Look for and make use of structure.

CCSS.MATH.PRACTICE.MP8 Look for and express regularity in repeated reasoning.

Notes

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