Abstract Reasoning with Trigonometric Functions and Their Inverses

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Rationale

I teach in a vocational school district consisting of four high schools that draw from middle schools in all five districts in New Castle County, Delaware. We use the *Core-Plus: Contemporary Mathematics in Context* (CPMP) textbook series, which is an integrated math curriculum. We teach math courses in 90-minute block periods every day for one 18-week semester; students have three to four courses per semester, including their chosen career area for 90 – 180 minutes per day for the entire school year. With some exceptions for the very high and very low achievers based on a district placement test, students take two semesters of *Core-Plus* (1 & 2) in their freshman year, *Core-Plus* 3 in their sophomore year, and a "Trigonometry" transition course in their junior year. I had been teaching a traditional precalculus course for seniors, and advanced juniors for several years. For a multitude of reasons beyond my control, I started the 2013 school year with a new precalculus course guide that followed the newly published *Core-Plus Mathematics: Preparation for Calculus, Course* 4. I had limited access to the new textbook prior to the start of the school year, but enough to be aware of major topics incoming students had never studied. Not surprisingly, we (all district teachers teaching this new precalculus course) found more holes as the semester went along. My goal for this curriculum unit is to patch some of those holes in the content with special attention given to CCSS Mathematical Practice Standard #2: Reason abstractly and quantitatively.

The most glaring hole came when I began teaching how to solve trigonometric equations. First of all, my students only had a crash course in graphs of trigonometric functions. Secondly, while students have used inverse operations to solve equations, they never learned about inverse functions. Fortunately, our course alignment has been adjusted so that both of these topics will be taught in prerequisite courses. However, my greatest challenge in teaching how to solve trigonometric equations is explaining why there are an infinite number of solutions and how to find and denote them based on the calculator's response to $\sin^{-1}$, $\cos^{-1}$, or $\tan^{-1}$ of some value. Over the years, even my most advanced Precalculus students have struggled with inverse trigonometric functions because these functions, being periodic, are not one-to-one unless the domain is restricted. Apparently I am not alone; in a *Mathematics Teacher* article, Elizabeth Teles wrote, "Many students, even after extensive study in precalculus of the inverse trigonometric function, fail to understand or appreciate the importance of the domain and range of these functions."[1]

I recently completed a curriculum unit through the 2014 YNI (Yale National Initiative) that set the groundwork for working with inverse trigonometric functions. In my previous unit, I focused heavily on domains and ranges when defining and working with functions. The unit covers operations on functions, including composition which requires the output of one function to be included in the domain of the other, finding inverses of functions, and cases in which the domain of a function must be restricted in order to be able to find an inverse. My goal for this curriculum unit is to extend my previous work on all types of functions and inverses to, specifically, trigonometric functions.

I am writing this curriculum unit for our district Trigonometry course, as well as for Precalculus. In the High School Functions domain of the Common Core State Standards for Mathematics (CCSS-M), "Trigonometric Functions" is its own section. This unit will address all of these standards to some extent. The first five of these standards are introduced in our Trigonometry course and built upon in Precalculus. Standards #5 through 9 are (re)taught in Precalculus. Standards HSF-TF.B.6 and B.7 are the most challenging for my students, so they will be the major focus of this unit. I believe the combination of my 2014 YNI unit and this one will give students a really firm understanding of all types of functions, including domain and range, inverse functions, and solving trigonometric equations.

Background Content

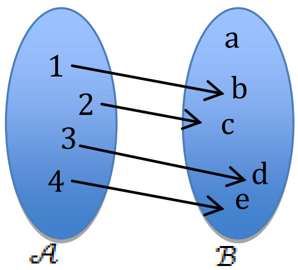
Required Prior Knowledge

I assume students have studied functions prior to beginning this unit. A function consists of three components: two sets and a relation between them. Set $\mathcal{A}$ is the set of all inputs, and is called the *domain* of the function. Set $\mathcal{B}$ contains the *range*, the set of all outputs. The *image* of a subset of the domain is a subset of the range. Set $\mathcal{B}$ is called the *codomain*. It may be larger than the image or range. For example, the codomain may be the set of all real numbers while the range contains only the interval [0, 10] for a given situation. A function can be represented in many ways as shown in Figure 1, and students should be fluent with all of them. In my previous curriculum unit "Using Math Practice Standards to Understand Functions and Their Inverses"[2] I emphasized the importance of considering the domain of a given function. In fact, changing the domain of a function changes the function, even if the function is represented by a formula. Take, for example, the function $f(x) = x^2$. For the domain $(-\infty, 0]$ the function is decreasing, but for the domain $[0, \infty,)$ the function is increasing. The formula is the same but the graph of the function is different.

Sets

$\mathcal{A} = \{1,2,3,4\}$ = domain

$\mathcal{B} = \{a, b, c, d, e\}$ = codomain



List of pairs (*input, output*)

(1, b), (2, c), (3, d), (4, e)

Table of Values

Input	1	2	3	4
Output	b	c	d	e

Words used to describe the association between the two sets.

Function Notation #1

$f: \mathcal{A} \to \mathcal{B}$

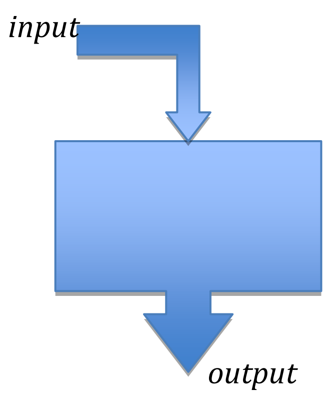
$f(1) = b, f(2) = c, f(3) = d, f(4) = e$

Function Notation #2

$\mathcal{A} \xrightarrow{f} \mathcal{B}$

$f(1) = b, f(2) = c, f(3) = d, f(4) = e$

Function Machine



Graph

$f: \mathcal{A} \to \mathcal{B}$ is the set of all pairs $(x, f(x))$ with $x \in \mathcal{A}$. The domain is read from the *x*-axis; the range is read from the *y*-axis.

Formula

$f(x) =$ any expression

Figure 1 – Function Representations

As part of students' study of functions, they were probably introduced to inverse functions. In past years, my students became adept at the process for finding an inverse function algebraically. Since the last time I taught inverse functions to students, I have learned a lot from my readings and by writing my last curriculum unit, and therefore plan to teach about Inverse Functions differently. I want students to use them in contextual situations to recognize the relationship between the domain and range in the function and its inverse. Based on what I learned in this seminar, I also want to emphasize the set of functions for which inverse functions exist, the operation performed on the set, and the role of the Identity element. Finally, I want students to have a deep enough understanding of inverse functions to be able to find inverses from graphs without knowing the associated formula.

The second prerequisite for this curriculum unit is the study of right triangle trigonometry. Students should be comfortable labeling the hypotenuse, opposite, and adjacent sides of a right triangle with respect to one of the acute angles. They should also be comfortable solving for unknown lengths and/or angles using Pythagorean theorem, or trigonometric ratios – sine, cosine, or tangent. It would be helpful to know the relationships for the 45°- 45°- 90° and 30°- 60°- 90° special right triangles, but not required.

Sets of Numbers

In our seminar on Algebraic Structure we learned the basics of set properties, equivalence relations and group structures. While I probably will not teach it directly to my students, it was helpful for me to see connections - similarities and differences - between different sets of numbers and how we perform operations on them. For example, there are some operations that are always possible when working with the set of Real Numbers that are not always possible when working with the set of Natural Numbers. Given two whole numbers, such as 3 and 5, the division operation is not possible because the quotient of $3 \div 5$ is not a Natural Number. However, the quotient does belong to the set of Real Numbers. I recently introduced Complex Numbers to one of my classes and gave them a (very) brief history of why we have the different sets of numbers that form the *Inclusion Statement:*

$$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R} \subset \mathbb{C}$$

$\mathbb{N}$ represents the set of Natural Numbers that includes counting numbers and zero. We need the number zero because "What do you have when you use all of your animals for food?" The need for the set of Integers, $\mathbb{Z}$, arose when people began borrowing from others and owed; they had negative values of things. Thus $\mathbb{Z}$ is the set of natural numbers and their opposites: $\mathbb{Z} = \{\ldots - 3, -2, -1, 0, 1, 2, 3 \ldots\}$. But what if we need to divide 5 bags of rice among 4 families? We need another set of numbers, Rational Numbers, that includes Integers and Natural Numbers and numbers that can be written as the ratio of two integers: $\mathbb{Q} = \left\{\frac{p}{q} \mid p, q \text{ are integers}, q \neq 0\right\}$. When Greeks worked with area, for example, they found numbers that could not be written as the ratio of two integers, as in the length of each side of a square field with an area of 5 m^2. (This is not the real history, but it makes sense to students, rather than telling them about mathematicians discussing the Pythagorean theorem for years.) The length $\sqrt{5}$ is an Irrational Number. Thus, there are both Irrational and Rational Numbers included in the set of Real Numbers, $\mathbb{R} = (-\infty, \infty)$. These are the numbers that students work with for the majority of their math careers. They don't think a lot about them; they don't even pay much attention when their graphing calculators show "ERR:NONREAL ANS" for the square root of negative numbers – that just means there is no solution to the problem! It isn't until they reach upper level math courses that they are introduced to Complex Numbers. The set of Complex Numbers, includes the *imaginary number, i,* defined as $i = \sqrt{-1}$. Then, Complex Numbers include all combinations of Real and Imaginary numbers: $\mathbb{C} = \{a + bi \mid a, b \text{ are real numbers}, i^2 = -1\}$.

Algebraic Properties and Operations

There are properties of operations that apply to some or all sets of numbers. Thinking about these properties and operations, as we did in our seminar, helps us formalize the algebraic structures we use in mathematics. Most high school students are aware of these properties, and use them, even if they do not know them by name. The Properties of Addition and Multiplication are 1) *Associative Property*: $(a + b) + c = a + (b + c)$ and $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all numbers *a, b, c,* 2) *Commutative Property*: $a + b = b + a$ and $a \cdot b = b \cdot a$ for all numbers *a, b, c* , 3) *Identity Property*: $a + 0 = a$ and $a \cdot 1 = a$ for all numbers *a*, 4) *Inverse Property*: $a + (-a) = 0$ and $a \cdot \left(\frac{1}{a}\right) = a$ for all non-zero numbers *a*, and 5) *Distributive Property* of multiplication over addition: $a(b + c) = ab + ac$ for all numbers *a, b, c*. These basic and well-known properties apply to other algebraic operations performed on functions, matrices, geometric transformations, and vectors. During our seminar meetings, I found it enlightening to review these properties with elementary teachers focusing on natural and rational numbers and see that these same properties apply to complex numbers, linear, exponential and trigonometric functions, polynomials and sequences.

Another property a set can have with respect to an operation is *Closure*: for a given set *and* operation, both the *input* and *output* of the operation are members of the set. Our seminar leader, Cristina Bacuta, defined an equivalence relation, $\mathcal{R}$, on a set, $\mathcal{S}$, as a collection of ordered pairs of elements of $\mathcal{S}$ that satisfy the Reflexive, Symmetric and Transitive properties. The most common equivalence relation is the equality relation for which these properties are as follows: a) *Reflexive Property*: *a* = *a* for any number in the set, b) *Symmetry*: if *a* =*b*, then *b* = *a*, and c) *Transitive Property*: if *a* = *b* and *b* = *c*, then *a* = *c*, As an example, consider the set of fractions and define the relation $\left(\frac{a}{b} = \frac{c}{d}\right)$ if and only if *ad* = *bc*. The fractions $\frac{3}{4}$ and $\frac{6}{8}$ satisfy the relation since $3 \cdot 8 = 4 \cdot 6$. Then, we see that $\frac{3}{4}$ is equal to itself $\left(\frac{3}{4} = \frac{3}{4}\right)$, demonstrating Reflexivity. Also, $\frac{3}{4} = \frac{6}{8}$ (since

$3 \cdot 8 = 4 \cdot 6$) and $\frac{6}{8} = \frac{3}{4}$ (since $6 \cdot 4 = 8 \cdot 3$) demonstrates Symmetry because of the Commutative Property of Multiplication. Finally, because $\frac{3}{4} = \frac{6}{8}$ and $\frac{6}{8} = \frac{12}{16}$ (since $6 \cdot 16 = 8 \cdot 12$) and $\frac{3}{4} = \frac{12}{16}$ (since $3 \cdot 16 = 4 \cdot 12$), the relation also has Transitivity. Therefore, the relation defined above is an *Equivalence Relation*, and all sets of equivalent fractions form "subsets of equivalent elements, called *equivalence classes*"[3] of rational numbers.

Algebraic Structures

Next, we learned about Algebraic Structures based on a set of numbers, functions or objects, etc. and one operation that takes two elements from the set, and after applying the operation, creates another member of the set. The least restrictive Algebraic Structure is called a *semigroup*. The two required properties for a set $\mathcal{S}$ and operation "$*$" to have a semigroup structure are i) closure: for any two elements, *a* and *b*, in $\mathcal{S}$, the result $a * b$ is also in $\mathcal{S}$, and ii) associativity: for any elements $a, b,$ and c in $\mathcal{S}$, $(a * b) * c = a * (b * c)$. An example of a semigroup is ($\mathbb{Z}$, +), the set of integers and addition. The sum of any two integers, such as -4 + 7 = 3, is also an integer, so there is closure. And since (-4 + 7) + -9 = -4 + (7 + -9) = -6, there is also associativity. However, ($\mathbb{Z}$, –), the set of integers and subtraction, is a counterexample because it lacks associativity: (-4 – 7) – 9 = -20 but $-4 - (-7 - 9) = 12$.

The second algebraic structure is called a *monoid*. Again, a set and operation must have i) closure and ii) associativity. The third requirement for a monoid structure is the existence of iii) Identity element: an element, *e*, in $\mathcal{S}$ such that $e * a = a * e = a$ for any element, *a*, in $\mathcal{S}$. An example of a monoid group is $(\mathbb{N}, \cdot)$, the set of natural numbers and multiplication. The product of any two natural numbers is also a natural number, so there is closure. Multiplication is also associative, and the identity element, *e* = 1, is an element of $\mathbb{N}$. As a counterexample, consider $(2\mathbb{Z}, \cdot)$, the set of even integers and multiplication. It has closure and associativity, but it does not contain an identity element (the multiplicative identity, 1, is not an even number). Therefore, $(2\mathbb{Z}, \cdot)$ has a semigroup structure, not monoid.

The third algebraic structure for a set and operation, a *group* structure, must have the same three properties as a monoid plus the existence of an Inverse element, *a'*. For any element, *a*, in $\mathcal{S}$, there exists an element, *a'*, in $\mathcal{S}$ such that $a * a' = a' * a = e$. In other words, performing the operation on an element and its inverse, in any order, returns the identity element, *e*. For example, $(\mathbb{Z}, +)$ has a group structure since every integer has an additive inverse that when added together equals the additive identity zero (e.g. -6 + 6 = 0). However, $(\mathbb{Z}, \cdot)$ does not have a group structure because the multiplicative inverse of an integer is not always an integer (e.g. $-6 \cdot -\frac{1}{6} = 1$, where $\frac{1}{6} \subset \mathbb{Q}, \text{not } \mathbb{Z}$), so the set is not closed under multiplication.

The final, most restrictive, algebraic structure we learned is an *abelian group*. A set S with operation $*$ has an abelian group structure if, like a group structure, it has i) closure, ii) associativity, iii) an identity element, iv) an inverse element, plus v) commutativity: for any elements, a, b in $S, a * b = b * a$. The set of integers with addition is an example of an abelian group, but integers with subtraction is not because subtraction is not commutative. There are additional algebraic structures for sets along with more that one operation, but they do not apply to this unit, so I will not consider them here.

Trigonometric Functions

Before researching this topic in preparation for writing this curriculum unit, I didn't think there was any difference between (right triangle) trigonometry and trigonometric functions. But, it turns out that trigonometry is a geometric concept that refers to the constant ratios – sine, cosine, tangent – of side lengths in similar triangles (all corresponding angle measures are congruent). On the other hand, trigonometric functions require an input (the domain consists of angle measures) to produce an output (the range is all real numbers). Because they are functions, each angle measure maps to a unique output for each trigonometric function. According to Mara G. Landers "trigonometric functions differ from functions that students have previously encountered in that trigonometric functions…cannot be expressed as algebraic formulae involving arithmetical procedures."[4] She explains that students must relate pictures of triangles to find numerical values of the trigonometric ratios. Moving to the unit circle connects the ideas of angle measures and right triangles "in ways that let (students) see trigonometric functions as relations between two quantities."[5] In fact, the definition of a trigonometric function given in the *Collins Dictionary of Mathematics* is "ratios of coordinates on the circumference of a circle centered at the origin, as radius, length *r*, sweeps out an angle, θ."[6] This definition explains why trigonometric functions are also called circular functions.

The Unit Circle

The *unit circle* is a circle drawn on a coordinate grid with its center at the origin, having a radius of one. For every point on the circle we can draw a radius to that point and a vertical line from the point to the *x*-axis to form a right triangle. The angle, θ, used as the input for the function, is measured counterclockwise from the positive *x*-axis, as seen in Figure 2. The input for the trigonometric functions can be any angle measure; the domain for sine and cosine is all $\mathbb{R}$eal numbers. After working with right triangles and always using the acute angles to set up and solve problems, this is a new concept for students. Now 90°, 180°, and even 0° angles have sine and cosine values! Furthermore, the sine and cosine values (the outputs) can be either positive or negative, or zero! However, before even forming right triangles, I will have students use protractors and rulers to find the *x*- and *y*-coordinates of points on the circle. Then they can form the right triangles and use the ratios to find the sine, cosine and tangent values of the same angle measures. (Refer to the Activities section for a more detailed description.) In a unit circle, since the hypotenuse is always the radius with a length of one unit, students should recognize that the $x = \cos(\theta)$ and $y = \sin(\theta)$. Even if the radius is not one, the sine and cosine ratios reduce to the same values because they are similar triangles whose lengths differ by a scale factor.

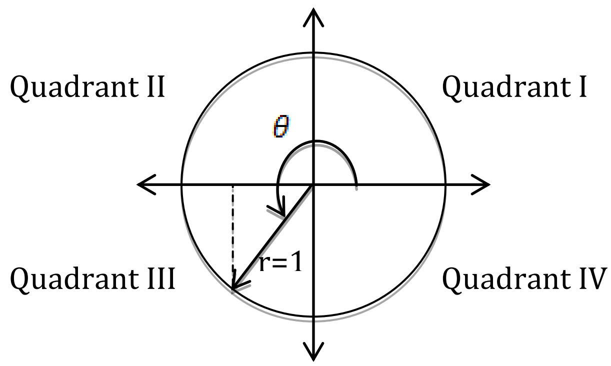


Figure 2 – The unit circle centered and the origin

I think spending time working with the unit circle from several different perspectives will be time well spent. As I stated above, measuring/finding the coordinates for points on the circle as the radius sweeps around the circle should help students see sine and cosine as functions of angles. They can also start to recognize that cosine (the x-coordinate) starts at a maximum value of 1 when $\theta = 0°$ and decreases to 0 when $\theta = 90°$. At the same time, sine (y-coordinate) starts at 0 when $\theta = 0°$ and increases to its maximum of 1 when $\theta = 90°$. When I ask students to find the tangent of the angle also, and then ask them if they recognize what the tangent represents, I always have a few students that recognize $\frac{opposite}{adjacent}$ as $\frac{y}{x}$, which gives the slope of the radius at that angle. In this way, they can see that the $\tan(0) = 0$ because the radius is a horizontal line; likewise $\tan(90)$ is undefined because the radius is a vertical line.

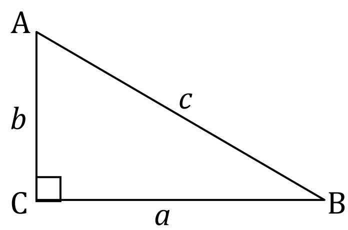
Another aspect of the unit circle is the symmetry. Any right triangle formed by a radius and a vertical line drawn to the x-axis from a point on the circle in the first quadrant can be reflected or rotated (i.e. rigid transformations) to form a congruent triangle in another quadrant. Therefore, the absolute values of sine, cosine, and tangent for angles greater than $90°$ are the same as those for the corresponding triangle in the first quadrant. The angle in the first quadrant is known as the *Reference* angle, and is the same as the acute angle formed by the radius and the x-axis in the other quadrants. Thus, the absolute values of $sin(30)$, $sin(150)$, $sin(210)$, and $sin(330)$ are the same. Students should see that cosine values (x-coordinates) are positive in the first and fourth quadrants, but negative in the second and third quadrants, based on the Cartesian coordinates of the points on the circle. Sine values (y-coordinates) are positive in the first and second quadrants, but negative in the third and fourth quadrants. Tangents (slopes) are positive in the first and third quadrants, and negative in the second and fourth. When students internalize the idea of trigonometric functions changing with angle measure, and can visualize the magnitude of angle measure, they will not need to memorize, nor depend on calculators, to determine the +/- sign of the function values. In that case, they can rely on what they already know about solving right triangles.

In their article, "The Circle Approach to Trigonometry," Kevin C. Moore and Kevin R. LaForest contend that students need a better understanding of angle measure to be able to use the trigonometric functions in novel contexts. They emphasize quantitative reasoning as a means of connecting angle measure and the sine function. Specifically they connect angle measures with partitions of an arc. They go on to state "radian measure is explicitly tied to arcs," and "degrees and radians measure the same quantity – that is, the "openness" of an angle – and are both used in trigonometry." [7] I will use some of these ideas when I teach radians to my students. My hope is that they will see radians as a proportional relationship to arc length (a fraction of the circle's circumference) rather than just as a conversion from degrees.

Finally, I will use the unit circle for students to create graphs of the trigonometric functions. Once they understand the connection between (x, y) and $(cos\theta, sin\theta)$ for points on a circle, students should be able to visualize the change in outputs as the angle changes, including positive and negative values. Also using the unit circle, students can see the periodicity of the trigonometric functions. They can visualize negative angles as going clockwise instead of counterclockwise, and angles greater than 360° as rotating more than one complete circle. It should help students identify the domain of sine and cosine functions as $(-\infty, \infty)$, or all real numbers, and the domain of the tangent as all real numbers except the odd multiples of 90° where it is undefined.

Trigonometric Equivalence Relations

The trigonometric function $f(x) = sin(x)$ is an *equivalence class* of functions. The relation, $\mathcal{R}$, defines two functions as equivalent if $f(x) = g(x)$ for all values of *x*. The domain of the set is all angles, $(-\infty, \infty)$. The range is all values of $sin(x)$, $[-1, 1]$. Let $g(x) = sin(x + 360) = sin(x + 2\pi)$. Because of its periodicity, $sin(x) = sin(x + 360) = sin(x + 2\pi)$. Another function in the class is $h(x) = cos(90 - x) = cos(\frac{\pi}{2} - x)$. Students can confirm this from what they know about right triangle trigonometry (Figure 3) or from graphs of $f(x) = sin(x)$ and $g(x) = cos(x)$ that show the two functions 90° out of phase (as a horizontal translation).



$m\angle A + m\angle B = 90° \rightarrow B = 90 - A$
$sin(A) = \frac{a}{c}, \cos(B) = \frac{a}{c}$
So, $sin(A) = cos(90 - A)$

Figure 3

We can confirm that the equality of functions is an Equivalence Relation on the set of functions defined from $\mathbb{R}$ to $[-1, 1]$. By using properties of trigonometric functions, we can find other functions in the equivalence class of $\sin(x)$. For all angles, *x*, $\sin(x) = \cos(90 - x)$ and $\cos(90 - x) = \sin(x)$, $\sin(x) = \cos(90 - x)$ and $\cos(90 - x) = \sin(x + 360)$, and $\sin(x) = \sin(\mathrm{x} + 360)$. As students study trigonometric identities, they will identify more functions in this equivalence class.

Inverse Trigonometric Functions

As discussed in the Background Content section above, a set and operation has a *group* algebraic structure if it has closure, associativity, and contains an identity element and an inverse element. To complete this unit, I will define the set, $\mathcal{S}$, as all *bijective* functions: $\mathcal{S} = \{\text{bijective functions}\}$ that have the same set as their domain and range, and the operation to be function composition. This set with respect to the composition of functions operation has a *group structure* but does not have an *abelian group* structure because it does not have commutativity; function composition is not always commutative. (Refer to my previous YNI curriculum unit "Using Math Practice Standards to Understand Functions and Their Inverses" for an in-depth discussion and practice of function composition.) Many of us learned the term *bijective* as "*one-to-one and onto.*" Simply stated, bijective means the function has different outputs for every different input (one-to-one), and *every* output corresponds to one input (onto) - each member of the range corresponds to a member of the domain with no extra outputs.

In my previous YNI curriculum unit, I provided practice identifying invertible functions from various formats: graphs, formulas, tables, and sets of points. I also included methods for finding and testing inverse functions, along with the need to restrict a function's domain in order to make it invertible. With the extended study of the unit circle I am including in this unit, I think my students will be better prepared to understand the need to restrict the domain for the trigonometric functions so that they have inverses. Furthermore, I think they will understand the accepted domain intervals. The graphs in Figure 4 illustrate the sine function with a domain of all real numbers for angles (I will assume radian measure) and a range of $[-1, 1]$. The function is clearly not one-to-one because there are an infinite number of angles with the same output value because of the function's periodicity. The highlighted area in each image indicates an interval that

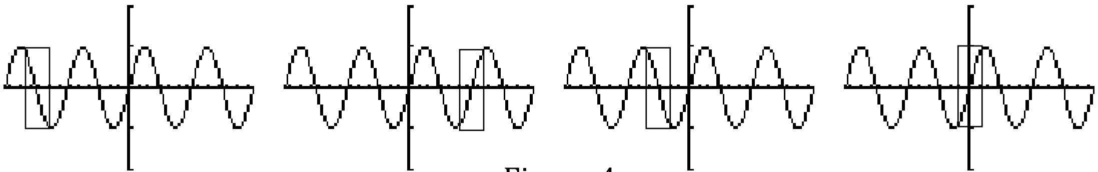


Figure 4

includes the full range of outputs *and* corresponds to a single input, making it one-to-one and onto. Because mathematicians like to include zero in a function's domain, the accepted interval for the restricted domain of the sine function so that it is invertible is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, shown in the fourth image. The cosine function is a horizontal translation of the sine function: $\cos(x) = \sin\left(x + \frac{\pi}{2}\right)$. Because of this shift, the cosine function is always decreasing over the interval $[0, \pi]$ and includes the full range of outputs, $[-1, 1]$. Therefore, the restricted domain of the cosine function so that it is invertible is $[0, \pi]$. Since the tangent graph has asymptotes at all odd multiples of $\frac{\pi}{2}$ (the slope of a vertical line is undefined, the *x*-value in the denominator is zero), the restricted domain interval of $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ includes zero, and produces the full range of outputs $(-\infty, \infty)$. The graphs in

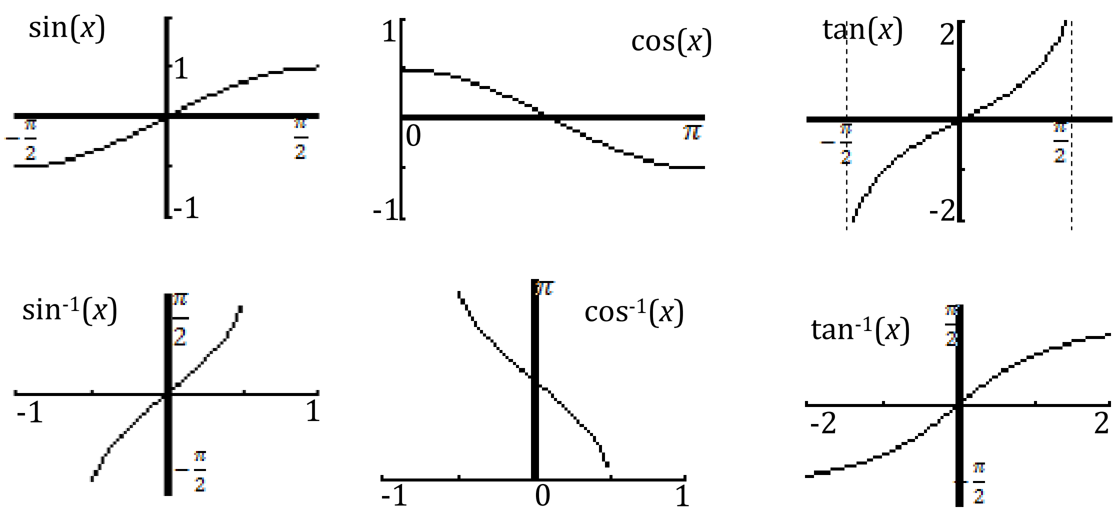


Figure 5

Figure 5 show the sine, cosine and tangent functions graphed with restricted domains and their inverses. In her article on "Equivalence Relations," Lisa Berger states, "the domain is determined by finding that set over which the original function is injective (one-to-one) and on the need to include a representative of every possible class of angle under some appropriate equivalence relation."[8] The equivalence relation makes the connection between *reference angles* (acute angles in the first quadrant) and all other angles having the same trigonometric values. This concept will be addressed again when solving trigonometric equations.

No Algebraic Structure

Let's consider a specific example of the set of bijective functions, $\mathcal{S}$, and the operation of function composition. If the domain and range are different sets, then the set has no algebraic structure because the operation of function composition is not always possible, or when the composition is possible some operation properties might not be satisfied for all functions. Here are some important examples to consider. An inverse function, f^{-1}, is a function that, when composed with f, in both directions gives the Identity, keeping the input value the same. It gets a little tricky here because the inputs and outputs of *f* and f^{-1} are different – angles versus function values – so there must be an identity element for each function. Suppose $y = f(\theta) = \sin(\theta)$, having the restricted domain $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ and range $[-1, 1]$, then $f^{-1}(y) = sin^{-1}(y)$, with domain $[-1, 1]$ and range $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. (Based on my readings for my previous unit, I intentionally use θ and *y* as the variables to help keep track of them; using the variable *x* for both the function and its inverse can lead to confusion, and errors.) Then $(f \circ f^{-1})(y) = \sin(sin^{-1}(y))$ takes an input on the interval $[-1, 1]$, maps it to an angle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, and maps the angle to its corresponding sine value, *y*, which proves that $(f \circ f^{-1})$ is the Identity function on $[-1, 1]$. In the opposite direction, $(f^{-1} \circ f)(\theta) = \sin^{-1}(sin(\theta))$ takes an angle on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, maps it to its corresponding sine value, *y*, then maps that value to an angle, θ, which proves that $(f^{-1} \circ f)$ is the Identity function on the interval $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$. Students will need to practice with problems, typically found in textbooks, such as "Evaluate $\tan(tan^{-1}(-1))$ and $cos^{-1}(\cos\left(\frac{2\pi}{3}\right))$."

Trigonometric Equations

My ultimate goal for this unit is for students to have a deep enough understanding of circular functions and their inverses to know how to find all solutions to trigonometric equations. For example, the equation $4cos^2\theta = 1$ becomes $cos\theta = \pm\frac{1}{2}$ using algebraic manipulations. Using $cos^{-1}\left(\frac{1}{2}\right)$, we get $\theta = \frac{\pi}{3} = 60°$, and using $cos^{-1}\left(-\frac{1}{2}\right)$, we get $\theta = \frac{2\pi}{3} = 120°$ from the calculator. However, the equation does not specify quadrants, or restrictions on θ. Students that understand reference angles and the symmetry in the unit circle will recognize that there are actually an infinite number of angles that are solutions

to the equation. The set of all solutions can be described using the set $\pi\mathbb{Z}$, which is the set of all real numbers of the form $n\pi$, where *n* is any integer.
This set under addition also has an abelian group structure. We can use this group to write the general solution to $4cos^2\theta = 1$ as $\theta = \{cos^{-1}\left(\pm\frac{1}{2}\right) + n\pi, n \in \mathbb{Z}\}$. It is important to remember that integers are both positive and negative, so adding $n\pi$ will include both positive and negative angles. The general solutions to linear trigonometric equations will be the set of angles plus n $(n \in \mathbb{Z})$ times the period of the function. The period for $f_1(\theta) = \sin\theta$ and $f_2(\theta) = \cos\theta$ is $2\pi = 360°$, and the period for $f_3(\theta) = \tan(\theta)$ is $\pi = 180°$. Note that the period of the function $cos^2\theta$ is π in the example above.

Activities

Lesson 1: The Unit Circle

Part 1 – Radians

I have been using an activity to introduce and define radian measure for so many years that I don't remember its source. Provide a pair of students with two protractors, two pipe cleaners, and copies of eight circles, with centers marked, each having a different radius (3cm – 10cm). Students first draw a radius anywhere in any one of the circles. Next, they use the pipe cleaner to measure the length of the radius by bending it at the appropriate point. Then, students use the pipe cleaner to mark the end of an arc beginning at the intersection of the radius with the circle that is the same length as the radius. They draw a second radius to the end of the arc, forming a central angle whose intercepted arc length is equal to the radius, and then measure the angle with the protractor. Students repeat the process with the rest of their circles, sharing the work on eight circles between partners. As students finish, have them record their angle measures on the board. It is not necessary for every student to record his/her data because it becomes clear that all angle measures are in the range 57-62°; students calculate the class average and typically find the average to be 57-58°. Once students get over their surprise that all of the angle measures are the same for all sizes of circles, except for measurement error, define the angle measure as 1 radian.

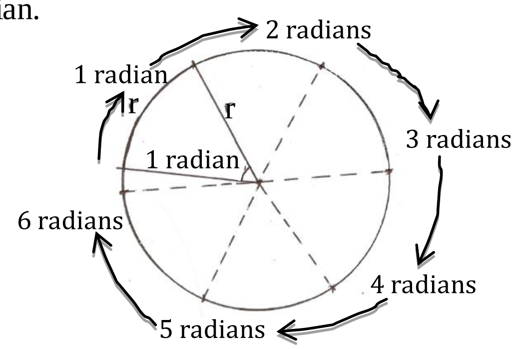


Figure 6 - Radians

The next part of the activity is to mark off the radians on the circumference of the circle using the pipe cleaner (Figure 6). Students that measure carefully find there are a little more than 6 radians in the circle; some estimate 6.2 or 6.3 radians. With a little bit of theatrics, I repeat, "Hmmm, 6.2 radians in a circle," until someone realizes it's 2π radians! Once they are convinced that there are 2π radians in a circle, reinforce the formula they know for the circumference of a circle: $C = 2\pi r$. The follow-up, to help students visualize radians as angle measures related to arc length, is to provide practice problems such as:

1. What is the central angle of a circle with radius of 3 cm that intercepts an arc length of 6 cm? [Answer: 2 radians]
2. What is the arc length intercepted by an angle of 2.5 radians in a circle with radius of 4 cm? [Answer: 10 cm]

Part 2 – Setting Up the Unit Circle

To set up the unit circle, begin with a Cartesian coordinate grid and a circle centered at the origin. To help students are uncomfortable with fractions, this semester I took time labeling angles measured in radians, formed between a radius and the positive x-axis as the radius is rotated counterclockwise (0 radians is the radius on the positive x-axis). After labeling π radians where the radius is on the negative x-axis, halfway around the circle, and 2π radians after one complete revolution, move on to multiples of $\frac{\pi}{2}$ radians. Point out that $\frac{\pi}{2}$ is $\frac{1}{2}$ of π, which is equivalent to 90°, or $\frac{1}{4}$ of a revolution. Then use the concept of unit fractions and simplifying fractions to show $2 \cdot \frac{\pi}{2} = \frac{2\pi}{2} = \pi = 180°$, $3 \cdot \frac{\pi}{2} = \frac{3\pi}{2} = 3 \cdot 90° = 270°$, and $4 \cdot \frac{\pi}{2} = \frac{4\pi}{2} = 2\pi = 360°$. Continue with multiples of other "special" angles: $\frac{\pi}{4} = \frac{1}{4}$ of $\pi = \frac{180}{4} = 45°$ and $\frac{\pi}{6} = \frac{1}{6}$ of $\pi = \frac{180}{6} = 30°$.

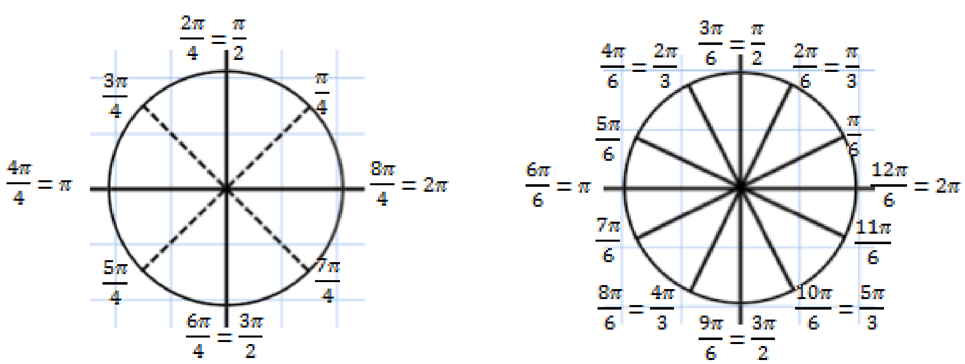


Figure 7

I recommend redrawing the circle (Figure 7) rather than adding the $\frac{\pi}{6}$'s because there are too many lines for students to see the equivalent divisions. Taking the time to illustrate radian measures as fractions of the circumference and using unit fractions to calculate the equivalent angle in degrees made my students more successful converting between radians and degrees. For example, to convert $\frac{13\pi}{10}$ to degrees, first find $\frac{\pi}{10} = \frac{180}{10} = 18°$. Then $13 \cdot \frac{\pi}{10} = 13 \cdot 18 = 234°$. Students should sketch a circle illustrating 10 divisions in each half $\left(\frac{\pi}{10}\right)$. In that way, they can confirm that $\frac{13\pi}{10} = 234°$ is in the 3rd quadrant. To convert from degrees to radians, students can calculate the fraction with respect to 180°: $220° = \frac{220}{180} = \frac{11}{9}$ which is $\frac{11}{9}\, of\ \pi = \frac{11\pi}{9}$. Alternatively, students can set up a proportion to solve for equivalent radians in a full circle: $\frac{220°}{360°} = \frac{x}{2\pi}$. Some students also think of the relationship as being a scale factor and apply it to 2π. In the end, students have a deeper understanding of radians because of the visual and numerical connections they make.

Part 3 – Connecting the Unit Circle to Right Triangle Trigonometry

This activity is based on one in the September 2008 *Mathematics Teacher* magazine.[9] The teacher creates a 10x10 Cartesian grid on graph paper and draws a circle centered at the origin with a radius of 10 blocks. Label 5 blocks on each axis as ± 0.5 and 10 blocks as ± 1, appropriately, so that the scale is 0.1 per block. In the activity, each student 1) uses a protractor to measure an angle from the positive x-axis, 2) draws the radius that forms the angle, and then 3) estimates the (x, y) coordinates of the intersection of the radius with the circle for angles $0 \leq \theta \leq 360°$; negative angles and angles greater than 360° can be added later. I created a table for students to collect and organize data:

Angle, θ	Quadrant	x-coordinate	y-coordinate	$cos\theta$	$sin\theta$
40°	I	~0.75	~0.65	0.77	0.64
…					

After measuring several angles in all four quadrants and recording positive and negative coordinate values, students then use their calculators to find the cosine and sine of each angle. It does not take long for them to realize that the x-coordinate is equal to $cos\theta$ and the y-coordinate is equal to $sin\theta$. At this point, I lead a class discussion about how the *unit circle* connects to what they know about right triangle trigonometry (refer to the Background section above). The discussion also includes recognizing $tan\theta = \frac{y}{x} = \frac{sin\theta}{cos\theta}$ as the slope of the radius after a rotation of θ.

Still using their unit circle, the activity has students computing the sine and cosine of $0°, 90°, 180°, 270°$ and $360°$ without protractors or calculators. This is a new concept for students since a radius at these angles does not form a triangle. Students seem accepting of these values, but I will revisit them when we graph the functions. The activity continues with questions about whether sine or cosine values for given angles will be positive or negative, forcing students to visualize the angle and its quadrant. Other questions ask students to compare the relative sizes of sine or cosine values of two angles, again by visualizing them without using calculators.

I added another part to the activity in the *Mathematics Teacher* article: Select an angle, θ, in the first quadrant and ask students to draw the radius rotated θ. Next, they should draw a vertical line from the intersection of the radius and the circle to the x-axis to form a right triangle. Let $\theta = 40°$, for example. The coordinates on the circle are (cos40, sin40) = (0.77, 0.64). Tell students to repeat the process for $\theta = 140°, 220°,$ and $320°$. When they realize that the absolute value of all the cosine and sine values are the same, lead a discussion about symmetry and congruent triangles in each quadrant that can be obtained by reflections (or rotation) of the original triangle. Define the *Reference Angle* as the acute angle formed by the radius and the x-axis (independent of quadrant). Students then practice finding the angles having equivalent reference angles in the other three quadrants, along with their cosine and sine values, given any angle.

Part 4 - Summarize

As a summary activity to reinforce the symmetry ideas, reference angles, radian measure and special right triangles, students complete a unit circle they can use for reference throughout the remainder of the course. I give them a template of a circle on a Cartesian grid with all multiples of $\frac{\pi}{6}$ radians drawn with solid lines, and all multiple of $\frac{\pi}{4}$ radians drawn with dashed lines as shown in Figure 7. Students also draw and label side lengths for $45° - 45° - 90°$ and $30° - 60° - 90°$ triangles at the bottom of the same paper, and use them to compute the sine and cosine values of $30°, 45°, \text{and } 60°$ in radical form. The completed unit circle contains angle measures in degrees and radians (and revolutions, if desired) on the "spokes" and the coordinates $(x, y) = (cos\theta, sin\theta)$ at the appropriate points on the circle. Once completing the angles in the first quadrant, the remaining coordinates can be filled in using the symmetry of the unit circle.

Lesson 2: Graphing Trigonometric Functions

Part 1 – Understanding the Shape of Sine, Cosine and Sine Functions

After all of the unit circle activities in the previous lesson, students understand why sine and cosine are functions – for any input from the domain, there is only one output in the range. They also understand that the domain for sinθ and cosθ is all real numbers since the radius can rotate in both a positive and negative direction an infinite number of times to form any angle. Students plot, by hand, all of the points (θ, sinθ) that they recorded in the table for the unit circle activity. These points include angles in all four quadrants, including ones on the *x*- and *y*-axes. In this way they see the value of sine begin with zero for $\theta = 0°$, increase (non-linearly) to a maximum of one for $\theta = 90°$, decrease past zero for $\theta = 180°$, then to a minimum of -1 for $\theta = 270°$ and back to zero after a complete rotation of $360°$. From their graphs students can visualize the cycle repeating in both directions to infinity. Class discussion should emphasize the fact that $\sin(\theta)$ is measuring the distance above $(+)$ or below $(-)$ the *x*-axis. Discussion should also confirm the range to be the interval [-1, 1], and can also be used to define *period* and *amplitude* for $f(\theta) = \sin(\theta)$. Students then repeat the process, graphing (θ, cosθ), to see that the value of cosine begins with its maximum of one for $\theta = 0°$, decreases (non-linearly) past zero for $\theta = 90°$ to a minimum of -1 for $\theta = 180°$, then increases again past zero for $\theta = 270°$ to its maximum of 1 after a complete revolution of $\theta = 360°$. Again, the cycle continues in both directions to infinity, the range is the interval [-1, 1], and the period and amplitude are the same as the sine function. The discussion centered on $f(\theta) = \cos(\theta)$ should emphasize the fact that cosine is measuring the distance left $(-)$ and right $(+)$ of the *y*-axis. The most obvious difference between the graphs of sine and cosine functions is the *y*-intercept. Some students recognize that the graphs have the same shape, but start at different places in the cycle. If a student brings it up, I do define this *horizontal translation* as *phase shift*, but I don't emphasize it.

I do not ask students to graph $f(\theta) = tan\theta$ by hand. Nor do I discuss transformations of the tangent function. However, I do project a graph of the tangent function, and connect the shape of the graph to the slope of the radius at different angles. I discuss the reason for vertical asymptotes at odd multiples of $90°$ where the radius is vertical – reinforcement of the fact that the slope of vertical lines is undefined. I also point out where (at what angles) the slope of the radius is negative, positive and zero.

Part 2 – Transformations of Sine and Cosine Graphs

Once students have graphed $f(\theta) = sin\theta$ and $f(\theta) = cos\theta$, they investigate the effects of the parameters A, B, and C on the graphs of $y = Asin(Bx) + C$ and $y = Acos(Bx) + C$ using their graphing calculators. They summarize their results in a chart with a side-by-side comparison of all of the transformations on sine and cosine functions. A template of the summary chart is given in Appendix B.

Lesson 3: Inverse Trigonometric Functions

Part 1 – Restricting Domains to Allow Inverses

In Lesson 3 of my previous curriculum unit, students learned the definition of one-to-one (*injective*) functions and practiced recognizing whether or not a function has an inverse. They also practiced finding inverses from formulas, tables and graphs for one-to-one functions. The unit ended with an activity in which students restricted domains of (primarily) quadratic functions to make them invertible. In this lesson, begin with a graph of $f(\theta) = sin\theta$ with $-720° \leq \theta \leq 720°$ and confirm that it is not one-to-one since several angles in the domain map to the same output in the range. Ask students, working in pairs or groups, to name a domain for which $f(\theta) = sin\theta$ has an inverse. Record all of the suggested domains and hold a class discussion about the advantages and disadvantages of each. As part of this discussion, define *bijective* functions, also known as "*one-to-one and onto*," as those which map each member of the domain to a different member of the range with no extra values. In the end, the class should define the domain for $f(\theta) = y = sin\theta$ as $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ with the range as $[-1, 1]$. Then $f^{-1}(y) = \theta = sin^{-1}(y)$ has domain and range of $[-1, 1]$ and $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$, respectively. Repeat the process for $f(\theta) = cos\theta$ and $f(\theta) = tan\theta$. The domain and range for $f^{-1}(y) = \theta = cos^{-1}(y)$ are $[-1, 1]$ and $[0, \pi]$, respectively and the domain and range for $f^{-1}(y) = \theta = tan^{-1}(y)$ are $(-\infty, \infty)$ and $\left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, respectively. Be sure to include discussion about the range of the tangent function not including the endpoints $-\frac{\pi}{2}$ and $\frac{\pi}{2}$ because it is undefined at these values.

Part 2 – Practice Finding Inverses

I developed a matching game to help students practice inverse trigonometric functions, especially with respect to the domain and range. The game uses a set of cards (Appendix C) with $sin^{-1}, cos^{-1}, tan^{-1}$ and θ values for special right triangles. The game requires students to form matches with three cards that make a true statement:

Function⁻	*value*	=	*angle*

There are 25 statements (using 75 total cards) that can be made. The activity can be done in groups of three or four students. Each group starts with 75 cards, and sorts them into three piles – function, value, and angle. One student in the group randomly chooses from the function pile, and a second student randomly chooses from the value pile. A third student either finds the matching angle, or determines that a true statement cannot be made. If there is a fourth student, he/she should record the complete statement, or the reason it cannot be true; if there is not a fourth student, the first student should record. Students should switch roles after each statement or reason is recorded. This game can also be done as a full class activity in which three students must get together to form true statements. This version will take more preparation for the teacher to be sure the cards in play will actually form true statements.

This game can be extended further by adding cards for *sin*, *cos*, and *tan*. Students can practice composing functions, and recognizing that a function composed with its inverse returns the Identity element. For example, $cos^{-1}\left(\cos\frac{\pi}{4}\right) = cos^{-1}\left(\frac{\sqrt{2}}{2}\right) = \frac{\pi}{4}$ and $cos\left(\cos^{-1}\frac{\sqrt{2}}{2}\right) = cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$. Problems such as $cos^{-1}\left(\sin\left(-\frac{\pi}{4}\right)\right) = cos^{-1}\left(-\frac{\sqrt{2}}{2}\right) = \frac{3\pi}{4}$ should spark discussion because the restricted domains for sine and cosine functions are different.

Lesson 4: Solving Trigonometric Equations

Once students internalize the ranges for the inverse trigonometric functions, and practice with the special triangle values, they will be prepared to interpret the outputs from their calculators. When faced with solving trigonometric equations, students can use their algebraic skills to get a solution. However, trigonometric equations require some thoughtfulness to ensure they have found all possible solutions. Solving problems given in context will help students determine how many solutions make sense. Karen Brown presents situations in her 2013 Delaware Teachers Institute curriculum unit entitled "Real-Life Applications of Sine and Cosine Functions."[10] For example, using a sine function that models tidal data, students can solve for the time of the next high tide (single solution) or for the times of all low tides for the next week (multiple, but finite number of solutions). If there is no context, students need to find angles in an equivalence class that also satisfy the equation, based on their knowledge of the symmetry in the unit circle. Students will need to practice writing infinite solutions in terms of an integer multiple of the period of the function. I recommend the following checklist to help students solve equations for all possible solutions:

1. Sketch a graph of the function. (Use what you have learned about transformations.)
2. Are there restrictions for the situation/context, or are there an infinite number of solutions?
3. What solution will the calculator give (what is the range of the function)?
4. What are the additional solutions, if applicable?

Appendix A: Addressing Common Core State Standards

HSF-TF.A: Extend the domain of trigonometric function using the unit circle.
- HSF-TF.A.1 – Understand radian measure of an angle as the length of the arc on the unit circle subtended by the angle.
- HSF-TF.A.2 – Explain how the unit circle in the coordinate plane enables the extension of trigonometric functions to all real numbers, interpreted as radian measures of angles traversed counterclockwise around the unit circle.
- HSF-TF.A.3 – Use special triangles to determine geometrically the values of sine, cosine, tangent for $\frac{\pi}{3}, \frac{\pi}{4}$, and $\frac{\pi}{6}$…

HSF-TF.A.4 – Use the unit circle to explain symmetry (odd and even) and periodicity of trigonometric functions.

HSF-TF.B: Model periodic phenomena with trigonometric functions.

HSF-TF.B.5 – Choose trigonometric functions to model periodic phenomena with specified amplitude, frequency, and midline.

HSF-TF.B.6 – Understand that restricting a trigonometric function to a domain on which it is always increasing or always decreasing allows its inverse to be constructed.

HSF-TF.B.7 – Use inverse functions to solve trigonometric equations that arise in modeling contexts; evaluate the solutions using technology, and interpret them in terms of the context.

Appendix B: Summary of Trigonometric Graphs

	y = Asin(Bx) + C	y = Acos(Bx) + C
Amplitude Maximum/minimum *y*-values Reflection across x-axis Midline (vertical translation) *y*-intercept period = $\frac{2\pi}{B} = \frac{360°}{B}$ frequency		

Appendix C: Matching Game – Inverse Trigonometric Functions

$\sin^{-1}$	0	0	$\cos^{-1}$	1
0	$\tan^{-1}$	0	0	$\sin^{-1}$

1	$\frac{\pi}{2}$	$\cos^{-1}$	0	$\frac{\pi}{2}$
$\sin^{-1}$	$\frac{\sqrt{2}}{2}$	$\frac{\pi}{4}$	$\cos^{-1}$	$\frac{\sqrt{2}}{2}$
$\frac{\pi}{4}$	$\tan^{-1}$	1	$\frac{\pi}{4}$	$\sin^{-1}$
$\frac{1}{2}$	$\frac{\pi}{6}$	$\cos^{-1}$	$\frac{\sqrt{3}}{2}$	$\frac{\pi}{6}$
$\tan^{-1}$	$\frac{\sqrt{3}}{3}$	$\frac{\pi}{6}$	$\sin^{-1}$	$\frac{\sqrt{3}}{2}$
$\frac{\pi}{3}$	$\cos^{-1}$	$\frac{1}{2}$	$\frac{\pi}{3}$	$\tan^{-1}$
$\sqrt{3}$	$\frac{\pi}{3}$	$\cos^{-1}$	$-\frac{1}{2}$	$\frac{2\pi}{3}$
$\cos^{-1}$	$-\frac{\sqrt{2}}{2}$	$\frac{3\pi}{4}$	$\cos^{-1}$	$-\frac{\sqrt{3}}{2}$
$\frac{5\pi}{6}$	$\cos^{-1}$	-1	π	$\sin^{-1}$
-1	$-\frac{\pi}{2}$	$\sin^{-1}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\pi}{4}$
$\sin^{-1}$	$-\frac{1}{2}$	$-\frac{\pi}{6}$	$\sin^{-1}$	$-\frac{\sqrt{3}}{2}$
$-\frac{\pi}{3}$	$\tan^{-1}$	-1	$-\frac{\pi}{4}$	$\tan^{-1}$
$-\frac{\sqrt{3}}{3}$	$-\frac{\pi}{6}$	$\tan^{-1}$	$-\sqrt{3}$	$-\frac{\pi}{3}$

Appendix D: Resources

Materials for Student Activities

Pipe cleaners
Protractors
Copies of circles with center marked, radius varying from 3 – 10 cm
3/8-inch grid paper with *x*- and *y*-axes marked, and circle centered at origin with r = 10 u
Handouts described in activities and/or appendices

Annotated Bibliography

Barrera, Azael. "Unit Circles and Inverse Trigonometric Function." *Mathematics Teacher* 108, no. 2 (2014): 114-19.
"A method to determine...inverse trigonometric functions directly from the unit circle."

Benson, Christine, and Margaret Buerman. "The Inverse Name Game." *The Mathematics Teacher* 101, no. 2 (2007): 108-12. 2007. Accessed August 13, 2014. http://www.jstor.org/stable/20876057.

Berger, Lisa. "Equivalence Relations Across the Secondary School Curriculum." *Mathematics Teacher* 106, no. 7 (2013): 508-12.

Brown, Karen. "Real-Life Applications of Sine and Cosine Functions." http://www.cas.udel.edu/dti/curriculum-units/Documents/curriculum/guide/2013/Thinking%20and%20Reasoning/units/13.02.01.pdf.
Unit written for Delaware Teachers Institute in 2013.

Landers, Mara. "Building Sinusoids." *The Mathematics Teacher* 106, no. 8 (2013): 592-598. 2013.

MacInnis, Maureen. "The Human Unit Circle." *The Mathematics Teacher* 107, no. 6 (2014): 480.

Moore, Kevin, and Kevin LaForest. "The Circle Approach to Trigonometry." *The Mathematics Teacher* 107, no. 8 (2014): 616-23.

Weber, Keith, Libby Knott, and Thomas Evitts. "Teaching Trigonometric Functions: Lessons Learned from Research." *Mathematics Teacher* 102, no. 2 (2008): 144-50.
Unit circle activity, using protractors and Cartesian grid, connects (x, y) coordinates to (cos θ,sin θ) values where the radius intersects the circle in all four quadrants.

Endnotes

[1] Elizabeth Teles, "Understanding Arcsin(sin(x)) and Arccos(cos(x))," *Mathematics Teacher* 85, no. 3 (1992): 198, accessed July 9, 2014, http://www.jstor.org/stable/27967565.
[2] http://teachers.yale.edu/curriculum/viewer/initiative_14.05.08_u
[3] Lisa Berger, "Equivalence Relations Across the Secondary School Curriculum," *Mathematics Teacher* 106, no. 7 (2013): 508.
[4] Mara G. Landers, "Building Sinusoids, " *Mathematics Teacher* 106, no. 8 (2013): 593.
[5] Kevin C. Moore and Kevin R. LaForest, "The Circle Approach to Trigonometry," *Mathematics Teacher* 107, no. 8 (2014): 617.
[6] E. J. Borowski and J. M. Borwein, *Collins Dictionary of Mathematics* (London: HarperCollins, 2002), 575.
[7] Moore and LaForest, "The Circle Approach to Trigonometry," 617-618.
[8] Berger, "Equivalence Relations," 512.
[9] Keith Weber, "Teaching Trigonometric Functions: Lessons Learned from Research," *Mathematics Teacher* 102, no. 2 (2008): 146.
[10] Brown, Karen. "Real-Life Applications of Sine and Cosine Functions." http://www.cas.udel.edu/dti/curriculum-units/Documents/curriculum/guide/2013/Thinking%20and%20Reasoning/units/13.02.01.pdf.

Curriculum Unit Title: Abstract Reasoning With Trigonometric Functions and Their Inverses

Author: Nancy Rudolph

KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Trigonometric functions are useful for modeling periodic phenomena.

ESSENTIAL QUESTION(S) for the UNIT

How does a function best model a situation?

CONCEPT A	CONCEPT B	CONCEPT C
The Unit Circle	Trigonometric Functions and Graphs	Inverse Trigonometric Functions
ESSENTIAL QUESTIONS A	**ESSENTIAL QUESTIONS B**	**ESSENTIAL QUESTIONS C**
How does the relationship that one revolution of the unit circle is equal to 2π radians and 360° help convert between radian and degree measures of angles? How do reflections and the values of sine, cosine and tangent for reference angles help find trigonometric values of angles in other quadrants?	How can a trigonometric function be defined given amplitude, period and midline of a periodic situation? How are transformations of trigonometric functions useful for modeling situations?	Why do the sine, cosine and tangent functions not have inverses from 0 to 2π? In what ways can these functions be altered so they do have inverses? How does the period of a function help identify all solutions of a trigonometric equation?
VOCABULARY A	**VOCABULARY B**	**VOCABULARY C**
Unit Circle, Ordered Pair Radian, Periodicity Arc Length, Special Right Triangles Trigonometric Function, Reference Angle	Cycle, Domain Period, Range Amplitude, Midline Vertical Shift/Translation	Inverse/Invertible Restricted Domain One-to-one and Onto (Bijective) Horizontal Line Test

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES

Materials: pipe cleaners (class set, one per student), protractors, 3/8” grid paper, copies of circles, charts and cards described or given in the Activities or Appendices.