Foundations for Modeling Functions: A Problem Solving Approach

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Introduction

Functions are one of the most commonly used topics in mathematics classrooms across the country. They are used to describe such linear problems as distance traveled, wages for hourly employees, and the calculation of sales tax. Functions are also used to describe exponential and logarithmic problems involving the increasing costs of college tuition, population growth, and decay of radioactive substances, the pH scale, earthquakes and sound intensity. Although functions are the building blocks for mathematics and describing the world around us, research from Bowling Green State University indicates that students as well as future mathematics educators view functions as simply a formula which requires an act of substitution.[1] Over the past few years, I have noticed that my students seem to have become excellent with their procedural skills in some situations, and yet in other situations their errors are clear evidence that they have a definite lack of conceptual understanding regarding the concept of even a linear function. It amazes me that some definition based problems completely baffle a calculus student. By discussing the basic foundation of this concept, and using problem based instruction, I hope to develop a student population with a sense of understanding for the one of the most fundamental concepts of our discipline. I will be presenting this unit as an introduction to my calculus course, a sort of review to begin the year.

Concept of Function in High School Mathematics Curriculum

This unit will be presented to students in an introductory calculus class; as such these students have had many experiences with the concept of a function. In algebra I, these students will have been exposed to the linear function or $f(x) = mx + b$. They will have made many tables and graphs and discussed the concept of a constant rate of change $\frac{\Delta y}{\Delta x}$. They will have been introduced to the concept of a function as a machine that assigns one number usually called the input with a unique value known as the output. Although they are exposed to the concept of a constant rate of change, because this unit is usually not embedded with other functions that do not have a constant rate of change for comparison, I believe the concept is not fully grasped by my students.

After successful completion of the algebra I curriculum, students move into the algebra II curriculum which reintroduces functions as a written rule. This written rule is defined to be a rule such that *for each x value in the domain, there exists a unique y value in the range*. This is their first introduction to the terms of domain and range. During this time students are also exposed to operations on functions such as adding, subtracting, multiplying, dividing and composing. My students seem to perform extremely well in these symbolic operations, however asking them to find $f(x)+g(x)$ given the graph of $f(x)$ and the graph of $g(x)$ becomes a bit of a problem. Here in lies the discrepancy between truly understanding the idea versus performing a more basic skill.

During this course, my students would have also studied the quadratic function $f(x)=ax^2+bx+c$ in great detail. They will have learned the effects of changing the values of *a, b* and *c*, and what effect these parameters have on the graph and how to solve quadratic equations. They will have modeled problems of maximizing area as well as projectile motion of a ball or diver using the quadratic function. In addition to the quadratic function, the students will have been exposed to piece-wise functions. A slight review of a piece-wise function will be useful for the flat tax rate problem later in this unit. Students will also have spent time during algebra II studying exponential functions. In the exponential unit, students study the basic characteristics of exponential functions $f(x)=ab^x$ and how they can be used to model exponential growth and decay. I believe that during this time it becomes completely relevant to compare and contrast the concept of linearity with the concept of an exponential function.

As students move into the pre-calculus curriculum, the concept of a function is reviewed along with basic operations on functions in the earlier part of the semester. As the course moves along the students are exposed to logarithmic functions $f(x)=\log_b x$ and their characteristics. They have learned the usefulness of logarithms to solve exponential equations and their basic graphs as inverses of the exponential.

Although my students have had ample opportunities to study the different types of functions, I wanted to put together a review unit that would lead them into studying calculus and the concept of the slope of a curve. The unit will be designed as problem posing rather than lecture.

An Overview of the High School and Students

My unit will be taught to two different sections of Honors Calculus at Conrad Schools of Science during the 2013-2014 school year. Most of the students in these classes have been relatively successful in their previous mathematics courses, but have not necessarily been in an honors mathematics class before. At Conrad, a student who has successfully completed Pre-Calculus or Honors Pre-Calculus may register for the Honors Calculus class. Note: this is not the AP Calculus class. Each student registered for the class is a senior and will be looking to be successful on their college placement exams. This unit will be used as an introduction to their course.

Conrad Schools of Science is a magnet school under the umbrella of the Red Clay Consolidated School District. It is located in the Wilmington area and draws students from New Castle to Newark. Students apply to Conrad by application, essay, interview, and science project. Each student receives a score on each admission measurement and all students meeting a percentage of a given total score have their name placed into a lottery system. As a result of the lottery system, we have a somewhat diverse group of academically performing students, meaning, that students with "A" averages have an equally likely chance in the lottery as students with a "C" average. Our school is operated on an A B block schedule, which means I will have 90 minutes with each class every other day. This block schedule is specifically suitable and helpful when using a problem-based approach.

Some of the students that will be in this particular class are students that were assigned to Conrad through their feeder elementary school prior to the magnet school transition and were given the opportunity to stay in the program. Each grade level houses approximately 150 students and I will have two sections of Honors Calculus with approximately 28 students in each section. Conrad has a 37% minority population with 33% of our students qualifying for free or reduced lunch services. About 94% of our current juniors passed the state assessment in mathematics.[2]

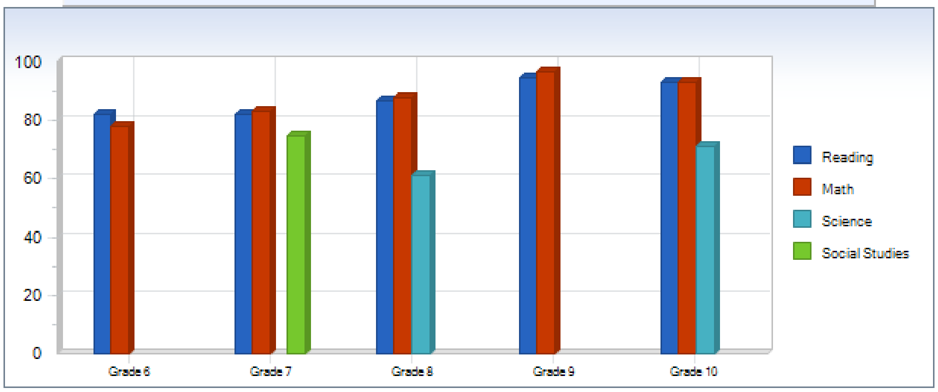
Enrollment by Race/Ethnicity

	2011-12	2012-13
African American	11.6%	10.4%
American Indian		0.1%
Asian	2.5%	2.7%
Hispanic/Latino	22.0%	20.3%
White	63.5%	66.0%
Multi-Racial	0.4%	0.6%

Other Student Characteristics

	2011-12	2012-13
English Language Learner	7.9%	7.9%
Low Income	33.9%	32.5%
Special Education	4.0%	3.0%
Enrolled for Full Year	100.0%	N/A

Percentage of Students Meeting State Standards (2012-13)



Unit Objectives

- Students will be able to define a function
- Students will be able to distinguish between situations that can be modeled with linear functions and with exponential functions.
- Students will be able to understand that linear functions grow by equal differences over equal intervals, and that exponential functions grow by equal factors over equal intervals
- Students will be able to recognize situations in which one quantity changes at a constant rate per unit interval relative to another.
- Students will be able to recognize situations in which a quantity grows or decays by a constant percent rate per unit interval relative to another.
- Students will be able to observe using graphs and tables that a quantity increasing exponentially eventually exceeds a quantity increasing linearly or quadratically
- Students will be able to interpret the parameters in a linear or exponential function in terms of a context.

The Unit Activities

An Overview

Linear Functions

Activity Number	Purpose	Time Frame
Activity 1 At the Pump	Warm up - Introduced the concept of a function based on prior knowledge.	20 minutes

Activity 2 Cup Song	Transitioned from Warm up into cup activity	5 - 10 minutes
Activity 3 Tower of Cups	Problem Solving using linear concepts	40 - 50 minutes
Activity 4 Function Questions	Warm up – posing questions on linearity using function notation	5 – 10 minutes
Activity 5 Copy Machine Problems and Extensions	Problem Solving in groups	40 - 50 minutes
Activity 6 Flat Tax	Homework Assignment	No time allotted in classroom

Exponential Functions

Activity	Purpose	Time Frame
Activity 1 Baseball Team	Introduce the concept of exponential function	20 – 30 minutes
Activity 2 Comparing Linear, Quadratic and Exponential Functions	To identify the similarities and differences of linear, quadratics and exponential functions	30 – 45 minutes
Activity 3 Pay Increase	Homework Assignment	No time allotted in classroom

Logarithmic Functions

Activity	Purpose	Time Frame

Activity 1 What is a Magnitude?	Introduction to logarithms as magnitudes	40 – 50 minutes
Activity 2 Creating a Log Table	Reinforce properties of logarithms	30 minutes
Activity 3 Logs of Other Bases	Extending beyond base 10	30 minutes

Linear Activities

1 At the Pump.

I used this activity as a simple warm up question that I posted on the board or delivered in a discussion environment. Building upon prior knowledge the idea is for students to start to see that in a function one variable is dependent on the other, by engaging them with something they already know. For example, gas is currently $3.15 per gallon; ask them to answer the questions

1. How much would it cost to fill your foreign-made vehicle which holds approximately 13 gallons? *$40.95*
2. How much would it cost to fill a large American SUV with a 21 gallon tank? *$66.15*
3. How much would it cost to fill a motor home with a 75 gallon tank? *$236.25*

Of course all students should have some frame of reference to become engaged in the problem and their answers will typically be exact. Next, I asked them to write an equation that would model this situation. How would they define their variables and which variable is dependent on the other? What represents the domain and what represents the range in this situation? This simple question and answer session got my students thinking and remembering the concept of a linear function.

2 Cup Song

The cup song was fairly popular this past summer with its connection to the movie Pitch Perfect.

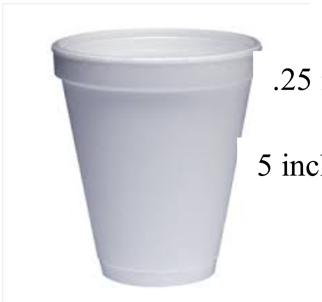
I played the i-Tunes version or the You Tube version of the song. My students recognized the tune and were curious as to what the next conversation will lead to. A few of my students wanted to demonstrate their skill on this song and I allowed them to entertain us for a few minutes. I even joined in. Remember this unit is my first unit for the school year; hence I am trying to make connections with my students. While the cup song was playing I distributed the materials for the needed for the next activity. (5 cups per group and a ruler)

3 Tower of Cups

In this activity, each group has received 5 Styrofoam cups and a ruler. The cups were stacked one inside of the other or nested.

The problem: I asked the students to determine the number of cups needed to stack in a nested format in order to match my height.

Of course the first question they asked was how tall I am. I told them my height was 5 feet 4 inches. A few asked me to convert my height to centimeters because they felt measuring the cup in centimeters was a better approach. Of course, I just gave them the conversion factor! As I walked around the room I listened to the conversations and asked questions. Most groups figured out there was an initial height and that a certain amount was being added each time. My actual cup height was approximately 121 cups. Most groups recorded answers between 120 and 130. Each group reported their findings and their techniques. We then discussed what type of function might best describe the stacking cup scenario and would this be modeled by a linear function? I began asking some questions of the entire class. I asked students to create a table. An example of a table

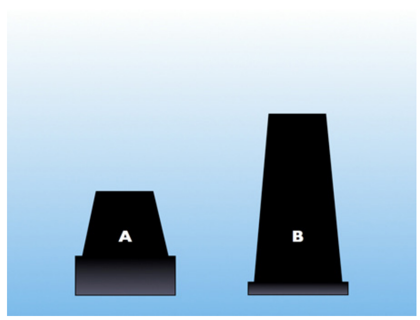


x: the number of stacked cups	y: the height of the stack
0	0
1	5.25
2	5.50
3	5.75
4	6.00
5	6.25
n	.25n + 5 provided n > 0

1. What was deemed important by your group? *The height of the base (from the bottom of the cup to the start of the lip of the cup) and the height of the lip*
2. What information related to the term y intercept or initial height? *The height of the base of the cup (from the bottom of the cup to the lip of the cup)*
3. What information related to the slope or rate of change? *The lip of the cup*
4. Can you identify the domain and range? *The domain is the number of cups stacked which would be a natural number, while the range is the height of the tower which would represent a non- negative real number*
5. Can you write the equation?
   height of a stack = (height of the lip)(number of cups) + (height of the base)
6. If I gave you a different cup with a larger base, what would change in the equation? *The initial height or the y intercept would increase*
7. What if the new cup had a smaller base? A larger lip? A smaller lip? *A smaller base would yield a smaller y intercept, a larger lip would increase my slope or rate of change, a smaller lip would decrease my rate of change or slope.*
8. How many parts of the cup are there? *Two*
9. Which part of the cup matters most over the long run? *The lip. The base only counts once but you count the lip every time.*
10. If I asked you to tell me how tall a stack of sixty cups would be, what would you do? *Add the height of sixty lips to the height of the base.*
11. If I asked you to go backwards and tell me how many cups are in a 200-centimeter-tall stack, what would you do? *Subtract the height of the base and then divide by the height of the lip.*
12. Does it matter if you round to the nearest centimeter? *Yes*

At this point at least one student raised their hand and questioned what happens when there are no cups? Explaining to me that if there are no cups then our height would be zero and our model would not work. A few students even realized it from the table as we began to construct our model. This is where I discussed limiting the domain of our function and how writing mathematical models does not always completely fit the world around us, but allows us to try to predict the future based upon our best model. I ask them if this could be defined as a piece-wise function. I pulled in the example of throwing a ball up into the air and what type of function is best at representing this situation. My students were quick to call out the parabolic function. However I asked them what is the height of the ball after 5 minutes in the air? They say the height is zero that by now the ball must have returned to the ground. Exactly I answered, but what does the parabola show us? With this small discussion, the students seem settled and accepting that our model is a good fit except for the starting value of zero, and thus we define our model for when $x > 0$.

After ample discussion concerning the cups, I put this graphic on the smart board and posed another series of questions



I asked for a volunteer to come to the board and measure the base and the lip of each cup. Then I asked them to determine

Which stack of cups will be taller after stacking three cups? *Cup B*
Which stack of cups will be taller after one hundred cups? *Cup A*
How many cups does it take before a stack of A cups will rise above a stack of B cups?

Now the students are really working!

4 Warm up - posing questions on linearity using functional notation

Let *f be a function such that whenever the input changes by 2, the output changes by 3.*
Does f represent a linear function? If so, identify the slope. If not, explain why

1. Let f be a linear function whose slope is 3

    a. Suppose $f(1) = 2$, what is $f(2)$? $f(5)$? *f(2) = 5, f(5) = 14*

b. Suppose $f(1) = a$, what is $f(2)$? $f(5)$? *$f(2)=a+3, f(5)=a+12$*

5 Copy Machine Problem and Extensions[3]

This activity was presented on the smart board and students worked in their groups. I used group names that I placed on 5x7 laminated cardstock. When a group finished a particular set of questions I asked them to place their card on the white board tray so that I could determine the number of groups that had completed the task. When 7 out of the 8 groups were finished, I pulled the class back together and we discussed their findings.

An office manager must decide between two options to fill the copying needs of his department. He wants to find the more economical option. AAA copiers, the first company contacted offers to lease a copy machine for a fixed fee of $50.00 a week and an additional charge of 2.1¢ for each copy.

For the same machine and comparable service, a second company, Speedy Print, offers a fixed charge of $180.00 a week with an additional charge of 0.5¢ for each copy.

Help the manager to make his decision.

The students were able to write linear equations to model each company. AAA was modeled by 50 + .021x, where x represented the number of copies made per week, while Speedy Print was modeled by 180 + .005x. The students were able to determine the break-even point (8125, 220.625) and that Speedy Print was more economical if a company planned on making more than 8,125 copies per week. Some of my students found the break-even point using a graphing calculator and other groups found it by setting the two equations equal to one another

Extension I

To entice customers during the summer session, AAA Copiers decides to eliminate its fixed charge of $50 a week. According to its advertisements, customers pay only for the copies they make!

When Speedy Print learns about the impending change, it immediately enters the price war by reducing its fixed charge, also by $50.00.

The office manager, having analyzed the original fee schedule, wants to know how these reductions affect the relative advantages of the deals available from the two companies.

My students quickly went into procedural mode. They wrote two new equations and began the process again. However a few of them began to realize that the only item that changed was the initial amount or y intercept and that they had changed by the same amount. Thus they reasoned the break-even point would have the same x value but different y value, since that represented the cost which would now be reduced by $50.00 (8125, 170.625)

Extension II
To entice customers during the winter session, AAA Copiers offers to lease a copy machine for a fixed fee of $50.00 a week and reduce the additional charge for each copy to 2.0¢.

When Speedy Print learns about the impending change, it immediately enters the price war by offering a fixed charge of $180.00 a week with an additional charge of 0.4¢ for each copy.

The office manager, having analyzed the original fee schedule, wants to know how these reductions affect the relative advantages of the deals available from the two companies.

Again a number of my students went directly into writing equations, but now I had a few more students' stopping to ask themselves what was happening and how did it affect the overall outcome. Students were starting to make connections between the break-even point and reducing each rate of change by equal amounts. They conjectured the number of copies would remain the same but that the cost would be reduced from the original scenario. (8125, 212.50)

Extension III
To entice customers during the winter session, AAA Copiers reduces the fixed fee to $30.00 a week and reduce the additional charge for each copy to 2.0¢.

When Speedy Print learns about the impending change, it immediately enters the price war by reducing the fixed charge into $160.00 a week with an additional charge of 0.4¢ for each copy.

The office manager, having analyzed the original fee schedule, wants to know how these reductions affect the relative advantages of the deals available from the two companies.

At this point the students knew the game! By reducing the initial amount by equal quantities and reducing the rate of change by equal quantities the students were able to see that the break-even point would still occur at (8125, 192.50)

6 The Flat Tax

The federal income tax laws are quite complicated. There have been various proposals to simplify the system. Many of these proposals are often referred to as a flat tax. To most people, a flat tax implies a single tax rate. This is usually not the case in most of the proposals. For example, a tax proposal endorsed by 1996 presidential candidate Steve Forbes proposed that a person would be taxed at a rate of 17% only on the income they earned above $13,300. The first $13,300 would not be taxed at all.

1. Why is Steve Forbes flat tax a function? Would it be a piece-wise function? *It would be a piece-wise function defined for when $x > 13{,}300$. For each value of x there would be a unique value for y. The amount of tax is dependent upon the amount of money earned.*

2. Give an equation for Steve Forbes flat tax where income is the input and the amount of the federal income tax is the output. What is the domain and range for your function?
t(x) = .17 (x – 13,300) , x > 13,300. The domain represents the earned income (all real numbers greater than or equal to 13,300) and the range represents the amount of federal taxes owed (all real numbers greater than or equal to zero)

3. How much would a person earning $20,000 per year pay in federal income tax? What percent of that person's income is used to pay federal income tax?
t(x) = .17(20,000 -13,300), thus taxes owed would be $1,139. Percent of income payable to federal taxes would be (1,139/20,000) x 100 = 5.695%

4. How much would a person earning $40,000 per year pay in federal income tax? What percent of that person's income is used to pay federal income tax?
t(x) = .17(40,000 -13,300), thus taxes owed would be $4,539. Percent of income payable to federal taxes would be (4,539/40,000) x 100 = 11.3475%

5. Does a flat tax imply that all workers will pay the same amount to the federal government? Explain
Students were surprised to see that a flat tax was not the same amount for everyone.

6. Give a formula for the function where the input is the income of an individual and the output is the percent of that income that goes towards the federal income tax. What is the domain and range of this function? Does this represent a linear function?
$p(x) = \frac{.17(x - 13{,}300)}{x} \bullet 100$ *The domain is the earned income (all real numbers greater than 13,300) while the range is the percent of income that is to be paid to the federal government for taxes (all real numbers between 0 and 100) This does not represent a linear function.*

Exponential Activities

1 Baseball Team

This activity was also presented on the smart board as a group activity and when most of the groups were finished with each situation the students shared their results and methods.

Suppose that as the owner of a baseball team who, with 10 games left in the regular season, is trying to sign a young player to a contract. The player convinces you to pay him in an unorthodox manner. Rather than a simple annual salary, the player asks you to give him one dollar for signing his contract and to double his salary each time the team plays a game. You, seeing a chance to secure a reasonably good player for what seems like an insignificant wage, readily agree. Given that the player will only receive $2 for the first game and there are only ten games left in the regular season, how expensive could this be?

How much did the player receive for playing 10 games?

Game	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*
Payment	*$1*	*$2*	*$4*	*$8*	*$16*	*$32*	*$64*	*$128*	*$256*	*$512*	*$1024*

Altogether the player received $2047 for playing the ten games.

What if the team made it into the final rounds and then ended up winning the World Series? (Playing the additional 19 games) what would you as the owner, owe the player for the last game?

The owner would owe the player $2^{29} = \$536{,}870{,}912$ *for playing the 29th game*

What would your total payout be for this player for all 29 games?

The total payout would be $1,073,741,823 for all 29 games

If this player participated in 10 regular season games and 12 playoff games before the World Series, how much would he earn for playing the first game of the World Series?

He would earn $8,388,608 for the first game in the world series.

This problem had enough detail and information for me to review the basic characteristics of exponential functions including the initial amount or y intercept and the growth factor. At this time I also asked the students to define the domain and range for this situation, with such questions as is -2 a member of the domain of this function?

2 Comparing and Contrasting Linear, Quadratic and Exponential Functions

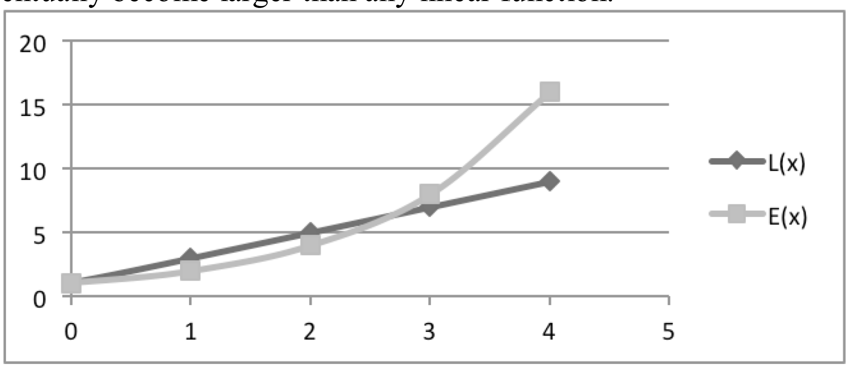
Exponential functions are similar to linear functions in that they both have a constant rate. However, what these rates measure is different and leads to different types of behavior. A linear function has a constant rate of change, $\frac{\Delta y}{\Delta x}$. This means that, every time the input increases by one, a fixed amount will be added to the previous output. For example, let L be the linear function where L(0) = 1 and the rate of change is 2 (i.e. the change in the output is twice the change in the input). Then the table gives the values of L for the first ten integers.

x	0	1	2	3	4	5	6	7	8	9	10
L(x)	1	3	5	7	9	11	13	15	17	19	21

An exponential function, on the other hand, has a constant growth factor. This means that, every time one is added to the input, a fixed amount will be multiplied by the previous output. For example, let E be the exponential function where E(0) = 1 and the growth factor is 2 Then the table gives the values of E for the first ten integers. It doesn't take long to see that multiplication produces a much faster increase then addition!

x	0	1	2	3	4	5	6	7	8	9	10
E(x)	1	2	4	8	16	32	64	128	256	512	1024

By examining the graphs of the two functions L and E, we can conclude that even though the linear function starts out larger for small values of x, the exponential function soon becomes much larger. A constant growth factor is significantly more powerful than a constant rate of change. In fact, any exponential function (with a growth rate larger than one) will eventually become larger than any linear function.



After examining the comparison of the linear and exponential graphs, I spent time looking at percent increases and decreases. I explained how to determine the growth rate based upon given percentages. Given a 7.3% increase indicates a growth rate multiplier of 1.073, while a 5.2% decrease would represent a decay rate multiplier of .948

To further the idea of the power of an exponential function, consider a problem on which a worker can choose how they would like to be paid.

Option 1: the worker would be paid $10 on day 1, $20 on day 2, $30 on day 3, $40 on day 4, $50 on day 5 and so on.

Option 2: the worker would be paid $1 on day 1, $4 on day 2, $9 on day 3, $16 on day 4 , $25 on day 5 and so on.

Option 3: the worker would be paid $1 on day 1, $2 on day 2, $4 on day 3, $8 on day 4, $16 on day 5 and so on

Which payments plan would you choose and why? By examining the three different functions by creating tables and equations, the students will quickly realize that the exponential function will eventually overpower the linear and the quadratic.

3 Pay Increase

An employee was offered the choice of receiving either a \$1000 bonus added to his base pay every year or a 2.7% cost of living increase added to his base pay every year. His current annual salary is \$33,500.

Let B be the function representing his annual pay if he chooses to take the \$1000 bonus option. Is B a linear or exponential function? Explain
The bonus option would represent a linear function. The initial amount would be \$33,500 and an additional \$1,000 would be added each year of service.

Find a model for B $b(x) = 33{,}500 + 1000x$

Let C be the function representing his annual pay if he chooses the 2.7% cost of living increase. Is C a linear or exponential function? Explain
The cost of living option would represent an exponential function, since the increase would be a percentage of the salary. It would increase 1.027 times per year.

Find a model for C $c(x) = 33{,}500(1.027)^x$

If he works for the company for five more years, which is the better choice?
If he works for the company for 5 more years, taking the bonus option would be the best

If he works for the company for twenty more years, which is the better choice?
If he works for the company for 20 more years, taking the cost of living option would be the best

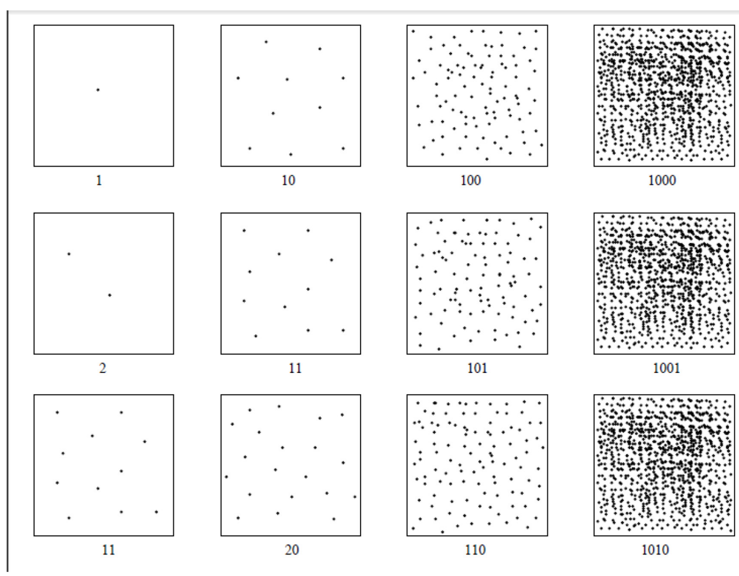
Logarithmic Activities

1 What is a Magnitude?

This year I decide to begin by defining logarithms as a magnitude. I have always introduced logarithms as in the inverse of the exponential, but this year I thought I would try a different approach. Using the dictionary definition of magnitude as "greatness of size", I began by discussing the metric system. Since the metric system is developed using a base of 10 we defined a kilometer as having a magnitude of 3 and a hectometer as having a magnitude of 2. As a class we decided to define the magnitude (m) of a number to be the power or exponent (b) such that $10^b = m$. From this definition we could define 100 as having a magnitude of 2 and 10 as having a magnitude of 1. Next I asked the students to try and find the magnitude of 50. Most students guessed and used a calculator to estimate the magnitude of 50. After that we created a table of values as such

b	1.4	1.5	1.6	1.7	1.8	1.9	2
$10^b = m$	25.12	31.62	39.81	50.11	63.09	79.43	100

By creating the table of values, students were able to estimate the magnitude of 50 and then examine the table to determine if the relationship might be modeled by a linear function. They were quick to decide that it would not be linear. From the magnitude approach we defined a logarithm to be the function where the input is a number and the output is its magnitude. From this definition we examined the graph of a logarithm and noted the graph was increasing and concave down. We determined the graph was increasing because the larger the number the larger its magnitude. We also noted that for larger values of x, the change in magnitude was less noticeable. This is shown in the charts below.



After discussing the concept of a magnitude, I reacquainted my students with the basic properties of logarithms.

2 Creating a Log Table

This question is posted on the smart board for students to work in their groups. Can you create a table of logs? The following table gives the logarithms for the first five integers. Answers are rounded to three decimal places.

x	1	2	3	4	5
log(x)	0	.301	.477	.602	.699

Use your knowledge of the properties of logarithms to find the logarithms of as many numbers as possible from 6 to 30. Note that you cannot use the values in the table to find log 7 since 7 is not the product of any number besides itself and one. Also 7 is not the quotient, root or power of any of the numbers given in the table.

As students try to develop their table, I circulated the room and listened to their discussions about properties of logarithms.

When students were completed and had a chance to share their results, I then asked them if they had to create the table from 1 to 100, which numbers could you easily find by knowing the logarithms of one through five. Are there any numbers that you cannot find? What are they? Why or why not?

3 Logs of Other Bases

This particular problem became a homework problem that was then reviewed and discussed in class.

Suppose we were to use a base other than 10 for the magnitude of our logarithm function. If we chose base 2, our magnitude would now represent powers of 2 rather than powers of 10. Can you answer the following questions?

1. What is $\log_2 32$? $\log_2 1024$?
2. Complete the table for $\log_2 x$

x	1	2	3	4	5	6	7	8	9	10
$\log_2 x$										

3. Is it no longer true in base 2 that increasing the input by a factor of 10 results in an increase of one in the output? Why not? How much do you have to increase the input before getting an increase of one in the output?
4. Plot the points from your table and connect them with a smooth curve. Compare this graph to our original base 10 graph. How are the two graphs similar? How are the two graphs different?

Appendix A: Standards

The following list of Common Core State Standards in mathematics will be directly addressed in this unit

Primary Content Standards

- CCSS.Math.Content.8.F.A.1 Understand that a function is a rule that assigns to each input exactly one output. The graph of a function is the set of ordered pairs consisting of an input and the corresponding output.
- CCSS.Math.Content.8.F.A.2 Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions). *For example, given a linear function represented by a table of values and a linear function represented by an algebraic expression, determine which function has the greater rate of change.*
- CCSS.Math.Content.8.F.A.3 Interpret the equation $y = mx + b$ as defining a linear function, whose graph is a straight line; give examples of functions that are not linear. *For example, the function* $A = s^2$ *giving the area of a square as a function of its side length is not linear because its graph contains the points (1,1), (2,4) and (3,9), which are not on a straight line.*
- CCSS.Math.Content.8.F.B.4 Construct a function to model a linear relationship between two quantities. Determine the rate of change and initial value of the function from a description of a relationship or from two (x, y) values, including reading these from a table or from a graph. Interpret the rate of change and initial value of a linear function in terms of the situation it models, and in terms of its graph or a table of values.
- CCSS.Math.Content.8.F.B.5 Describe qualitatively the functional relationship between two quantities by analyzing a graph (e.g., where the function is increasing or decreasing, linear or nonlinear). Sketch a graph that exhibits the qualitative features of a function that has been described verbally.

Secondary Content Standards

- N-Q.1. Number and Quantities, Quantities: Reason quantitatively and use units to solve problems. Use units as a way to understand problems and to guide the solution of multi-step problems; choose and interpret units consistently in formulas; choose and interpret the scale and origin in graphs and data displays.
- N-Q.2. Number and Quantities, Quantities: Reason quantitatively and use units to solve problems. Define appropriate quantities for the purpose of descriptive modeling.
- A-CED.1. Algebra, Creating Equations: Create equations that describe numbers or relationships. Create equations and inequalities in one variable and use them to solve problems. Include equations arising from linear and quadratic functions, and simple rational and exponential functions.
- F-IF.5. Functions, Interpreting Functions: Interpret functions that arise in applications in terms of the context. Relate the domain of a function to its graph and, where applicable, to the quantitative relationship it describes. *For example, if the function h(n) gives the number of person-hours it takes to assemble n engines*

in a factory, then the positive integers would be an appropriate domain for the function (Modeling Standard).

- F-IF.9. Functions, Interpreting Functions: Analyze functions using different representations. Compare properties of two functions each represented in a different way (algebraically, graphically, numerically in tables, or by verbal descriptions).

Primary Mathematical Practice Standards

While I believe all eight mathematical practices are applicable to this unit, these five are highlighted throughout the unit:

- MP-1. Make sense of problems and persevere in solving them.
- MP-2. Reason abstractly and quantitatively.
- MP-4. Model with mathematics.
- MP-6. Attend to precision.
- MP-7. Look for and make use of structure.

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[1] Bowling Green State University David Meel
[2] State of Delaware School Profiles
[3] Dr. Cai

Curriculum Unit Title: ***Foundations for Modeling Functions: A Problem Solving Approach***

Author: Michelle Northshield

KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Linear Functions, exponential functions and logarithmic functions

ESSENTIAL QUESTION(S) for the UNIT

How can linear, exponential and logarithmic functions be used to model problems?

CONCEPT A	CONCEPT B	CONCEPT C
Linear Functions	Exponential Functions	Logarithmic Functions
ESSENTIAL QUESTIONS A	**ESSENTIAL QUESTIONS B**	**ESSENTIAL QUESTIONS C**
What are the characteristics of a linear function?	What are the characteristics of an exponential function and how do they compare to linear and quadratic functions?	What are some characteristics of logarithmic functions and how can we use magnitude as a way to define logarithms?
VOCABULARY A	**VOCABULARY B**	**VOCABULARY C**
Rate of change, Initial amount Increasing, Decreasing	Exponential, Initial amount Rate of growth or decay Increasing, decreasing	Magnitude Base Properties of logarithms

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES

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