Fostering a Deeper Understanding of Exponential Relationships Through Problem Solving

Elizabeth Lancellotti

Introduction

I have taught exponential function units many times from different curriculums and to different grade levels, but one thing is pretty much always the same; my students do not like it and they do not understand it. In our 9th grade curriculum, the exponential functions unit follows the linear functions unit. I have noticed in the past that students do not seem to be able to solve problems involving exponential relationships in the same way they do with linear relationships. Their understanding of exponential equations is often limited to memorizing the form $y = ab^x$ where the a value is your “starting point” and the b value is your “scale factor”. My unit for the Delaware Teacher’s Institute will be a 9th grade unit on exponential relationships and functions. I would like to deepen students’ understanding by giving them more in depth problems that involve real world applications of exponential patterns and provide an opportunity to gain an in-depth understanding of the characteristics of exponential functions and expressions.

School and Classroom Background

I teach at Hodgson Vocational Technical High School in Newark, Delaware. Because we are a vocational school, using application in our teaching is very important. Therefore, in this unit, I constantly want to reinforce the material with real world examples that may even connect to specific shops in my school. One challenge I may face with 9th graders is, in our school, we get students who come from middle school in different districts, both public and private. Therefore, their math backgrounds and comfort levels can often vary. Therefore, I do not assume any prerequisite knowledge and I will try to offer extension activities for students who have a stronger background.

The course that I will be teaching is entitled Integrated Math I. It is assigned to the majority of 9th graders as their first mathematics course at Hodgson. We use the Core-Plus Mathematics textbook series[1]. For this course, students cover most of Book 1. The unit that involves exponential relationships is entitled Unit 5: Exponential Functions. In this unit, students learn to recognize exponential growth and decay and model these relationships with exponential equations. It seems to me that this unit does not give the students enough opportunities to explore where exponential patterns come from and apply exponential relationships to real world relationships. The section on the exponential properties is particularly weak. There is not enough opportunity for students to understand where the properties of exponents come from and why they work.

Prior to this unit, students have completed a unit on Linear Functions. It is helpful that this topic will be fresh in their minds, because I will consistently be comparing and contrasting linear and exponential relations. A solid understanding of linear functions is necessary prerequisite knowledge for my students. They need to know how to recognize a linear pattern and write a linear function to model a set of data. In addition, students need to know the meaning of exponents as an operation. For example, students must understand that $5^3 = 5 \cdot 5 \cdot 5 = 125$. They also need an understanding of scientific notation, though I plan on doing a brief review of this before the unit.

Objectives

I will have three main objectives for the unit. First of all, I will present exponential pattern in multiple ways and in various contexts so that students will be able to recognize and model real world exponential patterns. There are many real world situations that involve exponential relationships. Two of the most common situations are population and savings accounts. I find population to be a very interesting topic. I would like my students to investigate why population grows or decays exponentially. Why doesn't a population grow in a linear pattern for example? Examples involving money are always relevant to students because students always love money and want to know how they can make money. Again, we can investigate why savings grow exponentially. Why not linearly? What is the benefit of an account that grows exponentially? An understanding of these examples will help students' knowledge of exponential relationships. In addition, when real world examples of exponential functions are used, the parameters of their equations have much richer meaning. These situations are almost always covered in mathematics curriculums. But what are some other examples of exponential relationships? In my unit, students will have the opportunity to investigate on their own to find more real world examples of exponential patterns. In my experience, I have seen that once students are not involved in an exponential unit, they do recognize exponential patterns because they are expecting it and are in that context. By presenting exponential patterns in multiple ways and through multiple contexts, students will be more aware of these patterns and their characteristics under any circumstances.

My second objective will be that students will understand the characteristics of an exponential equation and will be able to make comparisons between different functions both in the exponential function family and in comparison to linear or other non-linear functions. As I mentioned before, basic exponential functions have two main components, a starting point, often denoted with the letter *a*, and a growth or decay factor, often denoted with the letter *b*. In my first objective, students will gain a better understanding of these values, but I would also like students to be able to compare equations with similar and/or different values. For example, what happens when two exponential equations have the same growth factor but different starting point? What are the similarities and differences between two such functions? For this objective, students will investigate topics like half-life, doubling time, and long-term behavior. I have always found the fact that half-life and doubling time do not depend on the starting value of a function to very eye opening and helpful in comprehending the characteristics of an exponential equation. We can also push the comparison of functions to comparing linear and exponential functions. Linear functions, like exponential functions, are also taught as having a "starting point" and a value that makes them change. So how do these types of functions relate? Because students often have a strong understanding of linear patterns and functions, a comparison between the two will be beneficial.

Finally, students will be able to manipulate exponential expressions in order to simplify and recognize equivalent expressions. This objective may be the hardest one for me to reach. I have taught the rules of exponents many times. Usually it includes the rules for multiplication, division, powers, negative exponents, and zero exponents. This part of the curriculum is often very dry and usually involves students memorizing the rules without any understanding of why they work or their connection to exponential patterns and equations. There is no real problem solving involved. In order to get students to understand these rules, I will allow them to investigate the rules on their own in order to discover them and see how they connect to the exponential patterns and equations they have learned about earlier in the unit. I will also have students try to find equivalent expressions by using the rules of exponents in order to simplify expressions. Students will then be able to not only find equivalence, but also justify their answer with the properties they have learned.

Connection to the Common Core State Standards

In the state of Delaware, we are beginning to implement the Common Core State Standards[2], which reinforce problem solving that I will use in my unit. Many of the standards directly connect to my objectives. The standards state that students should "choose and produce an equivalent form of an expression to reveal and explain properties of the quantity represented by the expression.[3]" The standards state that students should "create equations and inequalities in one variable and use them to solve problems.[4]" They state students should be able to "distinguish between situations that can be modeled with linear functions and with exponential functions.[5]" In addition, they state that students should "interpret the parameters in a linear or exponential function in terms of a context.[6]" I will make sure to keep these standards in my mind when designing the activities of my unit.

Content Topics

My unit covers the key concepts for a basic understanding of exponential functions while reinforcing problem solving along the way. The unit will cover characteristics of exponential patterns and real world examples, comparison of exponential and linear functions, and simplifying exponential expressions.

Characteristics of Exponential Patterns and Real World Examples

The unit starts with an introduction to exponential patterns. Exponential patterns are characterized by a constant multiplicative change. A pattern that doubles each time is an example of exponential change and something that students will be familiar with. Therefore, it is a good way to introduce exponential change. One classic example is the story of the inventor of chess, an Indian mathematician named Sissa or Sessa. The story goes that the ruler of the kingdom was so impressed with his invention that he offered Sissa the prize of his choice. Sissa decided that he wanted one grain of wheat for the first square on the chess board, 2 on the second square, 4 on the third square, and so on, doubling for each successive square. When the ruler finally realized that there was not enough wheat in the entire kingdom to pay this prize, the mathematician became the new ruler of the kingdom. This is an excellent example of how exponential patterns can start small and then get very large, very quickly. The number of wheat on each square can be shown as the sequence 1, 2, 4, 8, 16, … But it also can be shown as the sequence $2^0, 2^1, 2^2, 2^3, 2^4, ...$. The equation for the number of grains of wheat, W, on the nth square would be $W = 2^{n-1}$. The number of grains of wheat on the 64th square, which is the last square on the chessboard, would be $2^{64-1} = 2^{63} = 9.223372037 \times 10^{18}$. You can see by calculating this number that there is no way the ruler could pay off this prize. Since students are used to linear patterns, this result may be surprising and should hopefully start getting them into a different mindset about patterns of change. You can alter this problem by looking at a situation where you triple or quadruple the number of wheat on each square. For example, which of the following gets you the most grain of wheat on the final square: the original prize, tripling on a 6 by 6 chessboard, or quadrupling on a 4 x 4 chess board? The question comes down to which is the largest number, $2^{63}, 3^{35}, \ or \ \ 4^{15}$? It turns out that the original plan still gives you the largest number of grains of wheat on the final square.

In all of the previous examples, the starting value of the pattern was 1. It is important to understand how a pattern changes with a different starting value. For example, consider the following table of values.

Table 1: This table shows an exponential pattern starting at 5 and multiplying by two each time x increases by 1.

x	y	Process column

0	*5*	*5*
1	*10*	*5 x 2*
2	*20*	*5 x 2 x 2*
3	*40*	*5 x 2 x 2 x 2*

In order to understand the pattern, I have added a process column. We could go even further in the process column and show it as $5 \cdot 2^0, 5 \cdot 2^1, 5 \cdot 2^3$,.... Therefore the algebraic rule for this pattern is $y = 5 \cdot 2^x$. A common misconception is to show the algebraic rule as $y = 5 + 2^x$. I think by demonstrating some sort of process column, students will see why this does not work. Because you are constantly multiplying by some factor, the initial value must be multiplied by the rest of the equation, not added. At this point, we can generalize an exponential function as having an equation of the form $y = a \cdot b^x$, where a is the initial value when $x=0$ and b is a constant multiplier, often referred to as a growth factor or decay factor when $b < 1$.

Once there is an understanding of what an exponential pattern looks like as a table, graph, and equation, then it is time to investigate some real world examples that follow approximately exponential patterns. The classic example of exponential growth in the real world is population change. Now, of course, population does not grow perfectly linearly because there are so many factors that affect population, especially human population, like disease, war, natural resources, economy, etc. For instance, during World War 2, population growth was minimal in the United States, but at the end of the war, the country experienced what is known as the baby boom. However, it is true that population growth is much closer to an exponential pattern of growth than a linear pattern of growth. So why doesn't population grow at a linear rate? Lets consider a starting population of 500 of some species. It does not have to be a human population. One year later, maybe that population has grown to 700. Would you expect that the next year, the population would be 900 or maybe more or less? Because population change depends on the current population, you would not expect a population to grow at a constant rate. For example, if you have two populations, one of 500 and one of 700, living in identical environments, which population would you expect to have the larger population growth? It makes sense that the larger population of 700 would have the larger growth since there are more of the species to reproduce. This is the sort of thinking that I would like students to do in order to better understand an exponential pattern and how it differs from a linear pattern.

Now lets assume that this population that started out at 500 and grew to 700 after a year is, in fact, growing exponentially. What can we expect the population to be after year two? Students could probably predict that it should be more than 200 because a larger population will lead to more growth. In this problem, the additive increase of 200 is not as important as the multiplicative increase of 1.4. Because 200 is 40% of 500, the population has actually grown by 40% and that is a trend we can expect to continue the following year. Therefore, if your new population is 700, and 40% of 700 is 280, then we can expect the population to grow by that amount, resulting in a new population of 980.

Understanding percent growth is an important part of this unit because percent growth or percent decay is an example of an exponential pattern. However students may initial see percent growth as linear because you add an amount to your previous amount. Even though you are adding to your previous total, as with the 40% percent growth in the previous problem, the amount you add changes with each step since a percentage is based on your previous amount. Linear growth does not depend on your previous amount. The same amount is added at each stage regardless of how big or small your previous number is. It is important that students are able to see adding 40% as the same as multiplying by 1.4.

Another real world situation that involves an exponential growth is a savings account that grows by a percent increase. Savings accounts tend to grow exponentially because it gives the customer incentive to put more money in their account. Again, this demonstrates that a larger value for *y* leads to a larger growth. Lets say you put $1000 into a savings account that grows 3% annually, meaning that at the end of the first year you will have *$1000 + (.03x1000) = $1,030*. The following year, your account balance will again grow by 3% of the amount you started with. So at the end of the second year, you will have *$1,030 + (.03x$1,030) = $1060.90*. In G. Polya's How to Solve It, the author suggests that the first step in problem solving is understanding the problem.[7] Students need to be comfortable understanding and calculating a percent increase before they determine an algebraic model for the situation. You will notice, that the increase from the first year is not the same as the increase from the second year. During the first year, your savings account increased by $30, while in the second year, your savings account increased by $30.90. So while it might seem like this is a linear pattern because your adding on an amount, your additive increase is not constant so this is not a linear pattern. Just because it is not linear does not mean it must be exponential. How can we show that a percent increase is exponential? I think a process column and simplifying expressions can help illustrate this pattern as exponential. As shown below:

Table 2: This table shows the example of a savings account that starts with an initial deposit of $1,000 and increases by 3% every year.

x yrs	**y savings acct, $**	**Process Column**
0	1,000	
1	1,030	$1,000+1,000\cdot.03=1,000(1+1.03)=1,000\cdot1.03$
2	1,060.90	$(1,030+1,030\cdot.03)=1,030(1+.03)=1,000\cdot1.03\cdot1.03=1,000\cdot1.03^2$

3	1,092.73	$(1{,}060.9+1{,}060.9\cdot .03)=1{,}060.9(1+.03)=1{,}000\cdot 1.03^2\cdot 1.03=1{,}000\cdot 1.03^3$

This can help illustrate that the algebraic rule to represent this relationship is exponential and can be written as $y=1{,}000\cdot 1.03^x$. Therefore any savings account with an initial amount, M, that grows by a *p* percent annually, can be modeled by the exponential function $y=M\cdot(1+p)^x$, where *y* represents the amount in the savings account after *x* years. In this function, *p* is the percent growth as a decimal. I noted before that savings accounts give you an incentive to deposit more money in your account. If you consider an account that starts at $2000 and grows at 3%, then after one year you have *2,000x1.03 = $2,060*. Another way that may help students understand the 1.03 as a growth factor is that after each year, you have 103% of what you had the previous year. 103% as a decimal is 1.03. You have made $60 in a year as opposed to the person who deposited $1,000 who made $30 in a year. The gap between the two accounts will only continue to widen.

So far, all the exponential relationships that I have discussed have involved exponential growth. However, there are many examples of relationships that decrease, or decay, that also follow an exponential pattern. The difference between an exponential growth function and a decay function is that in form $y=a\cdot b^x$, $b > 1$ for growth and $0 < b < 1$ for decay. Two examples of exponential decay are the breakdown of radioactive materials and the breakdown of medicine in our bodies. Lets look at an example where a person takes 200 mg medicine that breaks down at a rate of 7% every hour. After an hour, the body has lost *200 x .07 =14 mg* of medicine, so there is *200 – 14 = 186 mg* of medicine left in the body. Unlike exponential growth where the change over the next hour would be more, here it is less because there is less medicine present so losing 7% will be a smaller decrease. In exponential decay, a pattern decreases more rapidly at first and then levels out over time. Modeling percent decrease is a little different than modeling percent increase. Again, a process column can help illustrate the pattern.

Table 3: This table illustrates the amount of medicine in the body of a person who takes a 200 mg dose of medicine that decays by 7% each hour.

x (hrs)	**y (mg of medicine**	**Process Column**
0	*200*	
1	*186*	$200-200\cdot .07=200(1-.07)=200\cdot .93$
2	*172.98*	$186-186\cdot .07=186(1-.07)=200\cdot .93\cdot .93=200\cdot .93^2$
3	*160.87*	$172.98-172.98\cdot .07=172.98(1-.07)=200\cdot .93^2\cdot .93=172.98\cdot .93^3$

Therefore, the function that models this pattern is $y=200(.93)^x$, where *y* is the amount of medicine left in the body *x* hours after the person has taken it. Therefore, if M milligrams of medicine are taken in a person's body that decays by p percent each hour, then we can write the function $y=M(1-p)^x$, where y represents the amount of medicine present in the body after *x* hours. Understanding that the decay factor here is .93 and not .07 is critical. Similar to the percent growth idea, students should think about the fact that after each time increment 93% of the medicine remains in the body not 7%. This will help them conceptually understand where the .93 comes from.

Comparison of Linear and Exponential Functions

Students often have a strong understanding of the properties of linear functions, which makes them a good comparison point for exponential functions. There are many different ways we can compare linear and exponential functions. For this comparison, we will look at $y=mx+d$ and $y=a(b)^x$ where *b > 0*. The domain and range of all linear functions is $(-\infty,\infty)$. For exponential functions, it is more complicated. The domain of exponential functions is $(-\infty,\infty)$, but the range depends on the value of a. If a > 0, the range of the function is $(0,\infty)$. If a < 0, you can imagine the graph being reflected over the x-axis, making the range $(-\infty,0)$. Notice that 0 is never a part of the range of an exponential function. That is because $0=a(b)^x$ does not have a solution since $0\neq(b)^x$ for any base *b*. Therefore, exponential functions do not have an x-intercept while all linear functions have exactly one when $x=\frac{-d}{m}$, with the only exception being the linear function f(x)=0 which lies on the x-axis. Both linear and exponential functions have y-intercepts that you can clearly see as part of their equation. The y-intercept of the linear function is *(0,d)* and the y-intercept of the exponential function is *(0,a)*.

We can also compare the end behavior of linear and exponential functions. The end behavior of a function tells us what happens in the long run. You can think of it as what happens on the ends of the graph. It describes the behavior of y as x approaches infinity and negative infinity. End behavior of linear functions depends on whether the value of m is positive or negative.

Table 4: The table below shows the end behavior of linear functions for positive and negative slopes both as x approaches infinity and as x approaches negative infinity.

Value of m	**As $x\to\infty$,**	**As $x\to-\infty$,**
+	$y\to\infty$	$y\to-\infty$
-	$y\to-\infty$	$y\to\infty$

The end behavior of exponential functions depends on the values of both of its constants a and b. Its end behavior is dependent on whether a is positive or negative and on whether b is greater than 1 (growth) or between 0 and 1 (decay).

Table 5: The table below shows the end behavior of exponential functions for growth and decay and for positive or negative y-intercepts.

Value of a	**Value of b**	**As $x \to \infty$,**	**As $x \to -\infty$,**
+	$b > 1$	$y \to \infty$	$y \to 0$
+	$0 < b < 1$	$y \to 0$	$y \to \infty$
-	$b > 1$	$y \to -\infty$	$y \to 0$
-	$0 < b < 1$	$y \to 0$	$y \to -\infty$

Another way we can compare linear and exponential functions is by modeling two data points using a linear and exponential pattern. Lets look at the points (0,10) and (1,15). If we assume the relationship between x and y is linear, then we can say that the y-intercept of the function is 10 and the rate of change is 5. This means that every time x increases by 1, y increases by 5. We can model this relationship with the function $y = 5x + 10$. If we now assume this relationship is exponential, then we can say that the y-intercept is 10 and the rate of change is 1.5. This means that every time x increases by 1, y is multiplied by 1.5, which is a 50 percent increase. We can model this relationship with the function $y = 10(1.5)^x$. A table allows us to even further compare the function.

Table 6: This table shows the contrast between a linear and exponential model of the pattern containing the two points (0,10) and (1,15).

x	**y (linear)**	**y (exponential)**
0	*10*	*10*
0.5	*12.5*	*12.247…*
1	*15*	*15*
2	*20*	*22.5*
3	*25*	*33.7*

I included a value of x = 0.5 in order to show that when 0 < x < 1, the linear is actually growing faster than the exponential, but once x > 1, the exponential function is greater than the linear and is distancing itself from the linear more and more as x increases.

We can do a similar comparison with a decreasing linear and exponential pattern using the data points (0,20) and (1,10). If we assume the relationship is linear, then the y-intercept is 20 and the slope is -10, since as x increased by 1, y decreased by 10.

Therefore the linear model for this relationship is y = -10x + 20. Lets now assume the relationship is exponential. The y-intercept is still 20, but the relationship is decaying by 50 percent because as x increased by one, the value of y was cut in half. Therefore the exponential model for this relationship is $y=20(0.5)^x$. The table below shows values of both functions.

Table 7: This table shows the contrast between a linear and exponential model of the pattern containing the two points (0,20) and (1,10).

x	y (linear)	y (exponential)
0	*20*	*20*
0.5	*15*	*14.142…*
1	*10*	*10*
2	*0*	*5*
3	*-10*	*2.5*

For the decay example, the linear function actually decreases faster than the exponential after $x > 1$, which is the opposite phenomenon than what happened with the growth example. This is due to the fact that exponential decay levels off over time, decreasing by a smaller amount as the value of y decreases. There are many other comparisons that can be done. The two examples I did had equal y-intercepts and the second point given was when $x = 1$. This simplified the problem significantly, but to challenge students you can give many other examples of any two random points and model them both linearly and exponentially.

Another comparison that I think is important is a comparison between exponential functions. For example, lets go back to the savings account example that grows at a rate of 3 percent annually. How would the savings account compare between a man who invested $500 to a man who invested $1,000? Here we are comparing two functions with the same growth rate, but a different y-intercept, $y=500(1.03)^x$ and $y=1{,}000(1.03)^x$. At the start, the savings accounts differ by $500, but over time, the difference in the savings accounts will increase. For example, after one year, the savings account that started with $500 will have $515, while the savings account that started with $1000, will have $1,030. The difference between the two accounts is now $515. The difference will increase more over time. One thing the two functions have in common is when the accounts double. This is because the equations $1{,}000=500(1.03)^x$ and $2{,}000=1{,}000(1.03)^x$ have the same solution since both can be simplified to $2=(1.03)^x$. This is not an equation I expect students to solve algebraically, but they can find its solution by graphing the system

$\begin{cases} y=2{,}000 \\ y=1{,}000\cdot(1.03)^x \end{cases}$ or the system $\begin{cases} y=1{,}000 \\ y=5{,}000\cdot(1.03)^x \end{cases}$ and finding its intersection using a graphing calculator. A common strategy my students use to find this solution is to look at a table of values. This will not be a particularly effective strategy because it is hard to locate from a table when y = 1,000 or when y = 2,000. Using the graph is more effective. The solution to both systems occur at approximately *x = 23.45*. Therefore, both savings accounts would double after a little over 23 years.

Properties of Exponents and Simplifying Exponential Expressions

In my unit, students do not solve exponential equations because that would involve introducing logarithms. However being able to simply exponential expressions using properties of exponents is one of my main objectives. The main properties of exponents that students must learn in order to simplify expressions are the multiplication rule, division rule, and the power rule. The multiplication rule states that exponential expressions with the same base that are multiplied together simplify as shown: $a^b\cdot a^c=a^{b+c}$. The common misconceptions I see students may with this rule is that they believe $a^b\cdot a^c=a^{b\cdot c}$or they believe they can simplify an expression with matching bases, like saying $x^b\cdot y^c=(xy)^{b+c}$. The second property of exponents students will learn is the division rule, which states that $\frac{a^b}{a^c}=a^{b-c}$. Again, the common misconception here is to state that $\frac{a^b}{a^c}=a^{\frac{b}{c}}$. The final key property of exponents is the power of powers rule, which states $(a^b)^c=a^{bc}$. The common misconception here is that $(a^b)^c=a^{b^c}$.

So how can we get rid of these misconceptions? In my unit, I will present many problems where students will have to problem solve on their own before being given the actual properties of exponents. By doing this, students will have a stronger understanding of exponential expressions and will not just be memorizing an algorithm. I can give my students the expressions $a^b\cdot a^c$, $\frac{a^b}{a^c}$, $(a^b)^c$, a^{-b}, $\frac{1}{a^{-b}}$ and ask them to find ways to simplify. It will be up to them to understand the problem, devise a plan, carry it out, and look back to review the solution, as Polya describes[8]. We can start with examples involving numbers that students can actually evaluate like $2^3\cdot 2^5=256$. How can we now represent 256 using an exponential expression with base 2? And what does this tell you about multiplying exponential expressions with like bases? Since $256=2^8$, we can start to recognize a pattern. Similarly, we can use examples for the division and power rule, such as $\frac{9^6}{9^4}=81$and $\left(3^2\right)^3=729$. Students will be doing the investigating on their own and then will have to sum up their findings and generalize the property in order to figure out the algorithm. This will give students a much stronger understanding of how these properties are derived.

Once students have derived the properties of exponents, we can begin to demonstrate why they work. It helps to write out exponential expressions in order to later simplify. For the multiplication rule, we can show $2^3\cdot 2^5=(2\cdot 2\cdot 2)\cdot(2\cdot 2\cdot 2\cdot 2\cdot 2)=2^8$. For the division rule, we subtract the exponents of like bases because certain a certain part of the expression cancels out do to the fact that *a/a = 1* for any real number *a*. Therefore, we can show $\frac{9^6}{9^4}=\frac{9\cdot 9\cdot 9\cdot 9\cdot 9\cdot 9}{9\cdot 9\cdot 9\cdot 9}=\left(\frac{9\cdot 9\cdot 9\cdot 9}{9\cdot 9\cdot 9\cdot 9}\right)\cdot 9\cdot 9=1\cdot 9^2=9^2$. The division rule can also help lead to an understanding of negative exponents. For example, if we examine the expression $\frac{a^3}{a^7}$, we can conclude according to the division rule that $\frac{a^3}{a^7}=a^{-4}$. We can also simplify this expression by writing it out, as follows

$\frac{a^3}{a^7}=\frac{a\cdot a\cdot a}{a\cdot a\cdot a\cdot a\cdot a\cdot a\cdot a}=\frac{1}{a\cdot a\cdot a\cdot a}=\frac{1}{a^4}$. Therefore $a^{-4}=\frac{1}{a^4}$ and in general, $a^{-n}=\frac{1}{a^n}$for any real number base, *a*, and integer exponent, *n*. Similarly, $\frac{1}{a^{-n}}=a^n$for any real number base, *a*, and real number exponent, *n*. This is an important rule of simplifying because simplifying an exponential expression usually involves getting rid of any negative exponents.

After grasping the properties of exponents, it is important that students are able to apply these properties to more complicated problems. Many times simplifying an expression involves using multiple properties and developing your own strategy for simplifying. Here is an example of a complex exponential expression: $\left(\frac{2a^7b^4}{a^2b}\right)^3\cdot\frac{3ab}{b^3}$. To simplify this, we can first simplify the expression in the parentheses:

$\left(\frac{2a^7b^4}{a^2b}\right)^3\cdot\frac{3ab}{b^3}=\left(2a^5b^3\right)^3\cdot\frac{3ab}{b^3}$. Next, we can apply the power of powers rule to distribute the cubic power to the parentheses: $\left(2a^5b^3\right)^3\cdot\frac{3ab}{b^3}=8a^{15}b^9\cdot\frac{3ab}{b^3}$. We can simplify the fraction using the division rule: $8a^{15}b^9\cdot\frac{3ab}{b^3}=8a^{15}b^9\cdot\frac{3a}{b^2}$Now we can

multiply across and use the multiplication rule for exponents: $\frac{8a^{15}b^9}{1} \cdot \frac{3a}{b^2} = \frac{24a^{16}b^9}{b^2}$.
Finally, we can use the division rule one more time to finish simplifying:
$\frac{24a^{16}b^9}{b^2} = 24a^{16}b^7$. Here, we have used many properties of exponents in a strategic manner to simplify a complex exponential expression. There are other valid ways to simplify this expression, but they will all lead to the same simplified version.

Activities

Activity #1

This activity explores the idea of population growth and why it makes sense that population grows in a more exponential pattern rather than linear. I will use a general case of putting fish in a pond and seeing how the population grows over time. I think this activity is a good one to do in groups because a lot of the questions are good for a group discussion or even a whole class discussion.

<u>Fish in a Pond</u>

There is a large pond located somewhere in the United States. It is an ideal environment for inhabiting fish. At its max, it can hold 30,000 fish. A scientist wants to monitor the growth in the population of the fish in the pond. He starts by putting 100 fish in the pond in January of 2012. He returns in January 2013 and finds that the population has reached 134 fish.

1. How many more fish were there in January 2013 than in January 2012?

2. Do you expect the population in January 2014 to around 168 fish, more than 168 fish, or less than 168 fish? Justify your answer.

3. Population often grows in an exponential pattern because it is growing by a percent of its starting population. What percent increase in fish occurred from the year 2013 to 2014?

4. Write an exponential function rule to model this relationship, where P represents the population t years after 2012.

5. Use your rule from #4 to predict the population of fish in the pond in the year 2030.

6. When does the exponential model predict that the fish population will reach the maximum allowance for the pond?

7. If the population did grow in a linear pattern, write a function model for the fish population where P represents the fish population t years after 2012.

8. According to your linear model, when would the fish population reach the maximum allowance for the pond?

9. Why is there a big difference between your answers to #6 and #8?

Activity #2

This activity is an introduction to exponential decay and it directly connects with the first activity in my unit. It uses the same context but it deals with a population that is decreasing instead of increasing. Students should think of examples of population decrease. For example, a lot of species are endangered and even extinct because of the effects of the surrounding environment, disease, and other species. The activity is again ideal for groups. Question #4 should be done in groups and then a whole class discussion before students move on to the next question.

<u>Fish in a Pond: Part 2</u>

It is the year 2050 and the large pond in the United States has now reached its maximum fish population of 30,000 fish. However, outside factors are starting to affect the population. There are predators that eat the fish and there is a disease spreading around pond that is slowly killing them off ☹. In the year 2051, there are only 28,200 fish left in the pond.

1. How many fewer fish are there in 2051 as compared to 2050?

2. What is the percent decrease from 2050 to 2051?

3. According to a linear model, what would the population be in 2052? According to an exponential model, what would the population be in 2052?

4. Why does it make more sense that the population decreases by a percent rather than at a constant linear rate?

5. Assuming the fish continue to die off in an exponential rate, write an algebraic model for the population, P, after t years since 2050.

6. What is the expected population in the year 2075?

7. When will the fish population be less than 2,000 fish? When will the fish population be less than 500 fish?

Activity #3

This activity is a comparison between linear and exponential relationships. For both linear and exponential relationships, two points is enough to determine the function rule. Therefore, we can use two *(x,y)* points in order to explore more in-depth a comparison between linear and exponential models. This activity is more of an independent practice or homework activity. It should be used once students have a firm grasp of what defines an exponential relationship and how to write an exponential function model.

Linear vs. Exponential Relationships

PART 1:

1. Use the table to answer the following:

x	*y*
0	5
1	10

a. Assume the relationship is linear and write an algebraic rule to model it. Call your function $f(x)$.

b. Construct a table of values from $0 \leq x \leq 10$ with $\Delta x = 1$.

c. When will the $f(x) = 100$? Explain how you got your answer.

d. Now assume the relationship is exponential and write an algebraic rule to model it. Call your function $g(x)$.

e. Construct a table of values from $0 \leq x \leq 10$ with $\Delta x = 1$.

f. When will the $g(x) = 100$? Explain how you got your answer.

g. Determine when $g(x) > f(x)$. Write your answer in interval form.

h. Determine when $f(x) > g(x)$. Write your answer in interval form.

2. Now answer the same questions a. through h. using the following table.

x	*y*
1	6,000
2	3,000

PART 2:

DIRECTIONS: For each table, determine whether the relationship is linear, exponential, or neither. If it is a linear or exponential relationship, write the equation to model it. For a bonus, determine the equation of any relationship that is neither linear nor exponential.

1.

x	*0*	*1*	*2*	*3*
y	*40*	*45*	*50*	*55*

2.

x	*0*	*1*	*2*	*3*
y	*40*	*80*	*160*	*320*

3.

x	*0*	*2*	*4*	*6*
y	*1*	*4*	*16*	*64*

4.

x	*5*	*10*	*15*	*20*
y	*100*	*115*	*130*	*145*

5.

x	*1*	*2*	*3*	*4*
y	*2*	*8*	*18*	*32*

6.

x	*0*	*1*	*2*	*3*
y	*81*	*27*	*9*	*3*

7.

x	*2*	*4*	*8*	*9*
y	*2,050*	*4,050*	*8,050*	*9,050*

8.

x	*5*	*6*	*7*	*8*
y	*100*	*50*	*0*	*-50*

Activity #4

This activity should be used as a group activity that allows students to discover some of the basic properties of exponents. Students could do this activity in pairs or in groups of 3 or 4 and then end with a whole class discussion where groups share the general rules that they have found. After this activity, students can then practice applying the rules to more complex exponential expressions.

<u>Can you generalize the Pattern?</u>

DIRECTIONS: For each exponential, write out the exponential powers the long way in order to try to find a shorter way to write the expression. Then, use your findings to try to generalize the pattern. Example: $2^3 \cdot 2 = (2 \cdot 2 \cdot 2) \cdot 2 = 2^4$.

1. Multiplication Rule:
   a. $3^5 \cdot 3^3 =$

      long way: ___________________________________

      short way:_________________

   b. $x^4 \cdot x =$

      long way: ___________________________________

      short way:_________________

   c. $4^4 \cdot 5^2 =$

      long way: ___________________________________

short way:_________________

What's different about this example?

d. General Rule: $a^m \cdot a^n =$______________________

2. Power to a Power Rule:

a. $\left(4^2\right)^3 =$

long way: ___________________________________

short way:_________________

b. $\left(x^5\right)^2 =$

long way: ___________________________________

short way:_________________

c. General Rule: $\left(a^m\right)^n =$______________________

3. Division Rule:

a. $\frac{3^5}{3^3} =$

long way: ___________________________________

short way:_________________

b. $\dfrac{x^7}{x^6} =$

long way: ___________________________________

short way:_________________

c. $\dfrac{8^4}{5^3} =$

long way: ___________________________________

short way:_________________

What's different about this example?

d. General Rule: $a^m \cdot a^n =$______________________

4. Power to a Fraction Rule:

a. $\left(\dfrac{3}{5}\right)^4 =$

long way: ___________________________________

short way:_________________

b. $\left(\dfrac{x}{y}\right)^3 =$

long way: ___________________________________

short way:________________

c. $\left(\frac{p^3}{q}\right)^2 =$

long way: ________________________________

short way:________________

What's different about this example?

d. General Rule: $\left(\frac{a}{b}\right)^m =$____________________

Conclusion

These activities are meant to help students deepen their understanding of exponential patterns, functions, and expressions. I have seen students struggle with these topics often in mathematics. I believe the difficulty occurs because students memorize the form of exponential functions and exponent rules and don't learn why they work or where they come from. A great way for students to strengthen their understanding is to connect the ideas of exponential functions to that of linear functions since often students do have a good grasp of why a relationship is linear and how to model it with an algebraic function. In order to simplify exponential expressions, students must understand why we are able to simplify an expression. Memorizing rules of exponents will not be beneficial for kids because they will simply forget them later on unless they have a conceptual background to go with it. My unit allows students the opportunity to use problem solving in order to understand exponential relationships.

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[2] "Mathematics > High School: Functions > Linear, Quadratic, & Exponential Models," Common Core State Standards Initiative, accessed November 1st, 2013, http://www.corestandards.org/Math/Content/HSF/LE
[3] Mathematics > High School: Functions > Linear, Quadratic, & Exponential Models."
[4] "Mathematics > High School: Functions > Linear, Quadratic, & Exponential Models."
[5] "Mathematics > High School: Functions > Linear, Quadratic, & Exponential Models."
[6] "Mathematics > High School: Functions > Linear, Quadratic, & Exponential Models."
[7] George Polya, *How to Solve It* (Princeton, Princeton University Press 1945), 5.
[8] Polya, *How to Solve It*, 5-6.

Curriculum Unit Title: Taxation in Modern and Ancient Societies: Integrating Math and Social Studies

Author: April Higgins

KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Delaware Social Studies Civics Standard 1a: Students will understand that governments have the power to make and enforce laws and regulations, **levy taxes**, conduct foreign policy, and make war.

CCSS.Math.Content.6.RP.A.3c Find a percent of a quantity as a rate per 100 (e.g., 30% of a quantity means 30/100 times the quantity); solve problems involving finding the whole, given a part and the percent.

Enduring Understanding: Governments require their citizens to pay taxes to achieve financial security.

ESSENTIAL QUESTION(S) for the UNIT

- Why do governments have the power to levy taxes?
- How are tax amounts calculated?

CONCEPT A	CONCEPT B	CONCEPT C
Taxation Today	Taxation in Mesopotamia	Calculating Tax Amounts

ESSENTIAL QUESTIONS A	ESSENTIAL QUESTIONS B	ESSENTIAL QUESTIONS C
Why do governments levy taxes? How are taxes levied?	Why did the governments of Mesopotamia collect taxes? How were taxes levied in Mesopotamia?	How are tax amounts calculated?

VOCABULARY A	VOCABULARY B	VOCABULARY C
income tax sales tax payroll tax levy	surplus specialization tribute	rate base tax amount decimal percent

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES

Delaware Content Standards and Clarifications Documents	Delaware Department of Education- http://www.doe.k12.de.us/infosuites/staff/ci/content_areas/socialstudies.shtml
Teaching Civics	Rationale for teaching civics- http://www.civicmissionofschools.org/educators Lesson plans and resources for teaching civics- http://www.icivics.org/ Lesson plans, videos, and books for teaching civics-http://new.civiced.org/