Mathematical Modeling of Motion

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Rationale

Modeling has two significant roles in the Common Core State Standards for Mathematics (CCSS-M). It is both a Standard for Mathematical Practice and a Standard for Mathematical Content entwined through all of the other six content standards. As we move towards full implementation of CCSS-M, students need more exposure and practice to become "mathematically proficient."[1] I continually look for applications for students to relate the math they learn in the classroom to situations they encounter outside, which is what most of the NCTM (National Council of Teachers of Mathematics) Process standards, and several of the CCSS-M Mathematical Practice standards demand. I have been teaching precalculus at a vocational-technical high school for the last 4 years, and had physics added to my schedule this year. Since nearly all of my precalculus students take physics, this curriculum unit will integrate the mathematical concepts of vectors, trigonometry, and linear and quadratic functions with the physics concepts of one- and two-dimensional motion. I am writing this curriculum unit for my physics students, primarily in 12^{th} grade; however, this unit could be incorporated into math courses with units on quadratic functions, trigonometry, and vectors, as well.

Traditional math courses offer real-life applications as word problems at the end of each chapter. While many examples are familiar to students, they rarely have the opportunity to see how formulas were derived. During this unit, students will drop, launch, kick, or throw objects and collect data to describe the path of their motion. They will blend physical experiments with the Mathematical Modeling Cycle. I view this unit as an ideal opportunity to demonstrate the integral connection between science and mathematics.

Within CCSS-M, the fourth Standard for Mathematical Practice is titled "Model with mathematics." It states, "mathematically proficient students can apply the mathematics they know to solve problems arising in everyday life, society, and the workplace…. By high school, a student might…use a function to describe how one quantity of interest depends on another. Mathematically proficient students who can apply what they know are comfortable making assumptions and approximations to simplify a complicated situation, realizing that these may need revision later. They are able to identify important quantities in a practical situation and map their relationships using such tools as diagrams, two-way tables, graphs, flowcharts and formulas. They can analyze those relationships mathematically to draw conclusions. They routinely interpret their mathematical results in the context of the situation and reflect on whether the results make sense, possibly improving the model if it has not served its purpose."[2] This standard describes what we learned in our seminar on Mathematical Modeling. Our seminar discussions also emphasized that mathematical models should have quantitative and predictive value if they are to be useful.

In addition to being a Mathematical Practice standard, Modeling is one of the seven High School Standards for Mathematical Content in the CCSS-M. As a content standard, "modeling links classroom mathematics and statistics to everyday life, work, and decision-making." Because real-world situations are not always the same, students have the opportunity to be creative as they gain expertise in the modeling process. Modeling is also a *process*. It is this process that we experienced in our seminar, and that I will demonstrate in this curriculum unit.

Background – Mathematics

The system to be modeled in this curriculum unit will be two-dimensional motion, specifically projectile motion. Students will also refer to previous experience with one-dimensional motion. Projectile motion is defined as the trajectory of an object somehow "launched" into the air. Once the object is launched, gravity is the only (and constant) acceleration acting on it. Students will measure both vertical and horizontal displacement of balls as a function of time under varying conditions, including launch velocity, launch angle, and initial height. They should recognize the parabolic flight of an object when they graph vertical displacement (height) versus time. They may or may not expect the graph of horizontal displacement (range) versus time to be linear, but they will certainly recognize it as a linear relationship once it is plotted.

My math students have a lot of experience with quadratic equations. In their Integrated Math II course (typically during their freshman year), they studied the effects of the parameters a, b, and c on the graphs of quadratic functions of the form $y = ax^2 + bx + c$. They began their investigation with the simplest form of a quadratic function, $y = ax^2$. They quickly recognized that a positive value of a produces graphs of parabolas that open upward, and negative values of a make them open downward. They discovered that larger (absolute) values of a make the parabola narrower, or increase/decrease more quickly. By exploring equations of the form $y = ax^2 + c$, Integrated Math II students found that the value of c is the y-intercept of the graph, the point where the parabola intersects the y-axis. In this special case, the parabola is symmetric about the y-axis, and the coordinates of the vertex (maximum or minimum) of the parabola are $(0, c)$. Adding the linear term to give the quadratic function the form $y = ax^2 + bx + c$, students saw that, while the c value remained the y-intercept, the vertical line (axis) of symmetry and the vertex shifted either left or right of the y-axis. They later learned that the b value can be used to determine the equation for the vertical line (axis) of symmetry $\left(x = \frac{-b}{2a}\right)$, but my experience with students in later courses tells me that this fact does not stick with them.

Integrated Math II students solved quadratic equations by looking for specific y-values in a table or on a graph in their graphing calculators. They practiced solving quadratic equations of the form $y = ax^2 + c$ algebraically by isolating x^2 and then applying the square root property to find two possible ($\pm$) solutions. They solved equations of the form $ax^2 + bx = 0$ by factoring a common factor, x, and setting each factor equal to zero. For example, if $a = 4$ and $b = -12$,

$4x^2 - 12x = 0$	
$4x(x - 3) = 0$	equivalent factored form using the Distributive Property
$4x = 0$ or $x - 3 = 0$	setting each factor equal to zero
$x = 0$ and $x = 3$	are the two solutions to $4x^2 - 12x = 0$

Integrated Math II students are also introduced to the mechanics of using the a, b, and c values from the equation $ax^2 + bx + c = 0$ in the Quadratic Formula to find two possible solutions. They see examples of quadratic equations with two, one, or zero solutions and can recognize the number of possible solutions by looking at a graph of the quadratic function.

Students in Integrated Math III (most often sophomores) also study quadratic functions. They build on the basic skills they learned in Integrated Math II. They learn to find the coordinates of the vertex of a parabola by first factoring expressions in the form $ax^2 + bx$, and $x^2 + bx + c$, when possible, then identifying the line of symmetry that is midway between these two "zeroes," or x-intercepts. They then substitute the x-value of the line of symmetry into the quadratic function to find its corresponding y-value. Students also learn that the line of symmetry is equal to $x = \frac{-b}{2a}$, the first part of the Quadratic Formula. The second part of the formula $\frac{\pm\sqrt{b^2-4ac}}{2a}$ then determines the distance from the line of symmetry to each of the "zeroes," the x-values that make the equation equal to zero.

Within this unit on solving quadratic equations, Integrated Math III students boost their skills for manipulating terms to get an equation into the correct form (i.e. equal to zero) to be able to solve it. They reinforce inverse operations and number sense in solving them. They are also expected to compare algebraic solutions to graphs of the original function(s). One of the contexts for these first two "forays" into quadratic equations is profit as a function of the price of tickets to an event: $P(x) = -50x^2 + 2600x - 22{,}500$. The components of this profit equation were derived as part of the unit's activities. Another context is projectile motion in which students are given an equation for height as a function of time: $h(t) = -16t^2 + 40t + 4$. However, students are given a weak explanation of where the parameters -16, 40 and 4 come from; they are told that -16 represents gravity, 40 is the initial velocity and 4, the starting height.

Students see quadratic equations again in either Integrated Math IV or Intermediate Algebra (typically junior year), and then for a fourth time in precalculus. They strengthen their factoring skills in these courses; in Integrated Math III, factoring trinomials is primarily limited to expressions/equations with a leading coefficient of one. In precalculus they learn about complex solutions to quadratic equations, rather than stating “no solution.” Precalculus students learn to “complete the square” as a method for solving quadratic equations. They also complete the square to find the vertex form of a quadratic function in order to graph it and to identify its domain and range.

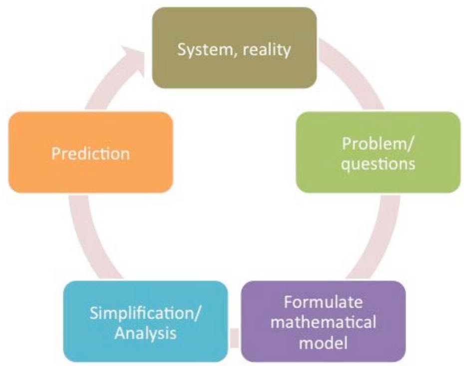
Trigonometry is taught in our Integrated Math III course. Students learn to find unknown measurements of side lengths and angles in right triangles and non-right triangles. Therefore, when they begin Physics, they have experience with the three basic trigonometric ratios, sine, cosine and tangent. They also have experience with the Law of Sines and the Law of Cosines. They have not, however, been introduced to vectors, and how to write velocity in terms of its vertical and horizontal components.

From my readings and personal experience in the classroom, students are always more engaged when they solve real world problems and collect and analyze their own data. The physical situation I described earlier will allow students to collect data for situations that are very real; all students have thrown and kicked balls and witnessed them flying upward and then downward. They will begin with the simplest situations (holding most variables constant) and then study the effects of changing more variables in order to create models that work in more complex situations and can predict outcomes to specific problems.

In this curriculum unit students will use apparatus (including adjustable ball launchers and/or catapults) and videos (pre-made and student-made), to collect data related to projectile motion. They will vary the parameters of initial height, initial velocity and launch angle to find their effects on the mathematical models (connecting trigonometry and quadratic functions). Students will use available electronic measuring device(s) and computer software to collect their data. They will use graphing calculators or other computer software to analyze their data and to write functions that can be used to predict further data points in both the vertical and horizontal directions. The ultimate goal is for students to be able to create a mathematical model that will help them predict the ideal position and conditions for launching a ball to hit a bulls-eye target.

Background – The Mathematical Modeling Cycle

In our early seminars, we discussed the meaning of Mathematical Modeling with respect to CCSS-M. We learned that there are two common ways that the word *modeling* is referenced in the literature. In the first definition, we begin with a math problem and build physical *models* for students to work with (aka, manipulatives) that help them understand the mathematics. In the second, we begin with a physical problem and build a mathematical model (e.g. a graph, table, diagram, or function) to explain the physics and predict future outcomes. In its simplest form, the mathematical modeling process is a cycle that is tested against reality and revised, repeatedly, until it is able to predict a real world situation accurately enough for our needs. Figure 1 illustrates the Mathematical Modeling Cycle we used in our seminar.



The Mathematical Modeling Cycle
Figure 1

We begin the process with a system or problem. Next, we identify variables that affect the situation. In the early stages, some variables that are considered less important should be held constant, or ignored. In that way, a simpler model can be developed, tested and revised until we can predict broad outcomes from it. More variables can be incorporated into revised versions of the model until it meets the requirements of the problem situation. Once an acceptable model is formulated we analyze it to draw conclusions, interpret the results in terms of the original situation, and finally, report on the conclusions or use it to predict future outcomes.

After working through several modeling examples that were accessible in different forms to students at all grade levels, K-12, we were able to categorize two different types of models. The first is a descriptive/empirical model that is derived from data. In this type of mathematical model, students collect data and summarize it using tables and graphs, and, at higher-grade levels, determine the best-fit line and function equation that describes the data. The second type of model is an analytical/fundamental model that is derived from "physical law." In this type of model, students use new or familiar concepts to understand and explain what they observe. They already have the mathematical tools (i.e. formulas, graphs) to describe the physical laws, and use them to answer questions or optimize the situation. In this curriculum unit, students will first derive an *empirical* model to describe projectile motion in the vertical and horizontal directions. In the end, students will use the concepts they learn to create an *analytical* model to satisfy specific conditions.

Since students already have experience with objects being thrown, hit or kicked into the air coming back down to the ground, they already understand the reality of the system. Therefore, after a brief launch discussion, they will move to the second step in the modeling cycle: *Problem/questions*. Students will brainstorm two things: the factors that affect projectile motion and questions about objects moving through air. At this stage, no questions are unreasonable. From the list, I will have students consider which questions need to be answered if their goal is to answer questions about motion. Also during this step, students will start to prioritize the key questions to make the process manageable; they cannot consider all factors at once.

Students will need to collect data for the third step in the modeling cycle: *Formulate mathematical model*. Depending on the equipment used, students may be able to see the data on a scatterplot instantly, or they can record data in tables and later use hand-drawn graphs, graphing calculators or computers to display it. From the graph, students will identify what type of model might fit the data best – linear, exponential, polynomial, or quadratic – and use technology to find the equation for the best-fit line.

In the fourth step: *Analyze the model/ Simplification*, students will discuss the usefulness and accuracy of their function equations. At this stage students will analyze whether or not there are other factors that affect the motion of the ball that they did not yet consider.

In the fifth step: *Prediction,* students will predict the answers to further questions such as "How long will it take for a heavier or lighter (more or less mass) object to reach the ground from a specific height?" or "What is the optimum angle and/or launch velocity to hit a target under given conditions?" They can then perform more experiments to answer more questions and revise their mathematical models, adding to the complexity of it with each repetition of the cycle.

Background - Physics

Describing One-Dimensional Motion

Early in any physics course, students study motion, beginning with one-dimensional motion. They learn that common measurements like speed and distance are not descriptive enough because these are scalar quantities (number values). Velocity and displacement are the corresponding vector quantities that have both direction and magnitude (size). Velocity is the rate of change in displacement over a time interval. The average velocity, v, is calculated using the formula $\boldsymbol{v} = \frac{d_2 - d_1}{t_2 - t_1}$ where $d_2 - d_1$ is the change in displacement for the time interval $t_2 - t_1$. When position is graphed versus time (Figure 2), velocity is seen as the slope of the graph. It is positive when the

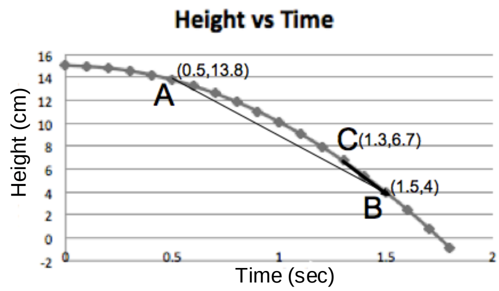


Figure 2

object is traveling in the positive direction ("positive" can be defined according to the situation), and negative when traveling in the opposite direction. For nonlinear graphs, the average velocity is the slope of the straight line connecting two points on the curve; the smaller the time interval, the more accurately the velocity can be calculated. In Figure 2, the velocity for the object traveling from point A to point B is $v = \frac{4-13.8}{1.5-0.5} = -9.8$ m/s. The velocity is negative because the object's displacement is in the downward direction. The straight line representing the slope between points A and B is only an approximation of the slope because the line does not exactly follow the shape of the curve. The slope for the smaller time interval between points C and B would give a closer approximation of velocity because the straight line connecting the points very nearly follows the curve: $v = \frac{4-6.7}{1.5-1.3} = -13.5$ m/s. Instantaneous velocity, at one point in time, is the slope of the line tangent to the curve at that point. Using calculus, velocity is calculated as the derivative of a position-time function and can be evaluated at any point in time. The function graphed is $h(t) = -4.9t^2 + 15$. Its derivative, $h'(t) = -9.8t$, evaluated at point B is $h'(1.5) = -9.8(1.5) = -14.7$ m/s.

During the brainstorming period, I would expect some students to name mass (or weight) as a factor affecting motion. They will be able to test the effect of mass by dropping objects of varying mass from a constant height and measuring the time it takes to reach the ground. Provided air resistance is not a factor, all objects should reach the ground in the same amount of time.

Next, physics students learn about acceleration. Acceleration is also a vector quantity, and is the rate of change in velocity over a time interval. The average acceleration, a, is calculated using the formula $a = \frac{v_2 - v_1}{t_2 - t_1}$ where $v_2 - v_1$ is the change in displacement for the time interval $t_2 - t_1$. On a graph of *velocity versus time*, acceleration is seen as the slope of the line. Positive acceleration means the velocity is increasing during the time interval, and negative acceleration (i.e. deceleration) means velocity is decreasing. Instantaneous acceleration is the slope of the graph at a specific time (the slope of the line tangent to the curve at that time for nonlinear graphs). Using calculus, the acceleration is the derivative of a velocity-time function, and can be evaluated at any point in time. Acceleration is also the second derivative of a position-time function. The acceleration for the function graphed in Figure 2 is constant and calculated as $h''(t) = -9.8$ m/s^2, the accepted value for acceleration due to gravity.

Cars traveling on a flat surface are an example of one-dimensional motion in the horizontal direction. For cars traveling at a constant rate (velocity), acceleration is zero, and the displacement will equal $d = vt$. For cars traveling at an increasing or decreasing rate, the displacement can be calculated by

$$d = v_{av}t = \tfrac{1}{2}(v_i + v_f)\cdot t = \tfrac{1}{2}[v_i + (v_i + at)]\cdot t = \tfrac{1}{2}(2v_i + at)\cdot t = v_i t + \tfrac{1}{2}at^2.$$

For projectile motion, there is only an initial velocity, with no acceleration other than gravity, so the horizontal distance traveled depends only on velocity and time the object is in the air. I expect my students to be surprised by this (because it was counterintuitive to me) but be able to verify it through experiments.

An example of one-dimensional motion in the vertical direction is free-falling objects, without the effect of air resistance. Falling objects are subject to constant acceleration due to gravity, g; the accepted value of g is –9.80 m/s^2 or –32 ft/s^2. It is my hope that my students will be able to verify these values of acceleration due to gravity in their experiments in this unit by creating graphs from data that look like Figure 2.

The same displacement and velocity relationships apply for vertical motion as for horizontal motion, except that the value of g is used in place of a. For example, an object dropped has an initial velocity, v_i, of zero and the displacement calculation becomes $d = ½gt^2$. If the object is shot straight up, $d = v_i t + ½gt^2$. The displacement is not equal to the total distance traveled by the object because it is a vector having direction. The displacement of the object in the upward (positive) direction is cancelled by the same displacement in the downward (negative) direction for a total displacement of zero.

Vectors

Vectors will be a new concept for my students. Vectors have a direction and a magnitude, and are represented in diagrams with arrows. For simplicity, we can place the tail of at least one of the vectors at the origin on a coordinate grid. The length of the arrow represents the magnitude of the vector. The angle formed by the vector with respect to the positive x-axis, gives its direction. Horizontal vectors (angles equal 0° or 180° from the x-axis) can be added by putting the tail of the second vector at the head of the first. The sum is then the total length of both vectors lying end-to-end. Likewise, vertical vectors (angles equal to 90° or 270° from the x-axis) can be added by placing them head-to-tail and finding their total length.

The fun begins when vectors are neither horizontal nor vertical. In that case, the vector can be split into its horizontal and vertical components using trigonometry. In right triangle trigonometry, the sine of an angle ($\sin\theta$) is defined as the ratio of the length of the side opposite the angle to the length of the hypotenuse (the magnitude of the velocity vector itself): $\sin\theta = \frac{opposite}{hypotenuse} = \frac{v_y}{v}$. Therefore, the vertical component of the velocity, as shown in Figure 3, is $v_y = v \cdot \sin\theta$. The cosine of an angle ($\cos\theta$) is defined as the ratio of the length of the side adjacent to the angle to the length of the hypotenuse: $\cos\theta = \frac{adjacent}{hypotenuse} = \frac{v_x}{v}$. Therefore, the horizontal component of the velocity, as shown in Figure 3, is $v_x = v \cdot \cos\theta$. If we know both the horizontal and vertical components, we

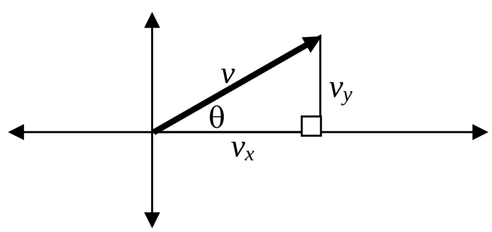


Figure 3

can calculate the magnitude of the velocity vector using the Pythagorean Theorem: $v^2 = {v_x}^2 + {v_y}^2$; v is the positive square root of v^2. We can also find the angle, θ, formed by the velocity vector and the x-axis if we know both the horizontal and vertical components using the arctan (inverse) function: $\theta = \tan^{-1}\left(\frac{v_y}{v_x}\right)$. When adding vectors that are not all horizontal or vertical, we still put them head-to-tail and then draw the *resultant* vector from the origin (also the tail of the first vector) to the head of the last vector. The horizontal component of the resultant is the sum of all horizontal components, and its vertical component is the sum of all vertical components. The magnitude of the resultant is then found using the Pythagorean Theorem and its direction, θ, (relative to the x-axis) using the arctan function defined above.

Describing Two-Dimensional Motion

Projectile motion, circular motion, periodic motion and harmonic motion are all examples of two-dimensional motion. In all types of two-dimensional motion, the horizontal and vertical components are considered independently. In projectile motion (the focus of this unit), the path of an object launched into the air is called the trajectory. We can describe the horizontal component by two parameters - its initial velocity and angle of release. We describe the vertical component by four parameters - its initial height, initial velocity, angle of release, and acceleration due to gravity. The independent variable for both components is time of flight, and the dependent variable is the displacement vector in either direction.

The horizontal displacement of projectile motion follows a linear path. That is, a graph of the horizontal displacement versus time is a straight line having a slope equal to the horizontal velocity component. Linear functions are written in the form $y = f(x) = mx + b$, where m is the slope and b is the y-intercept. Students can relate to this relationship because they have learned the formula $d = rt$, where distance is equal to the product of rate and time. A common context is a car traveling at a rate of 60 miles per hour for two hours will cover a distance of 120 miles. In terms of a projectile, we would write the equation as $x = v_x \cdot t$, where displacement is equal to the product of the horizontal velocity component and time (notice that mass is not a parameter in this function). The horizontal velocity remains constant throughout the object's flight (refer to Figure 3).

The vertical displacement of projectile motion follows a parabolic path; its trajectory can be described by a quadratic function. Quadratic functions are written in the form $y = f(x) = ax^2 + bx + c$. As I stated earlier, my students have studied quadratic functions in previous courses and generally recognize the effects of the parameters a and c. In projectile motion, the quadratic function takes the form $y = ½gt^2 + v_yt + h_0$, where g is the acceleration due to gravity (g = -9.8 m/s^2, or -32 ft/s^2), v_y is the vertical component of the initial velocity and h_0 is the initial height (notice that mass is not a parameter in this function, either). The vertical velocity decreases as the object rises, equals zero at the vertex, and then increases in the negative direction until it reaches the ground. Figure 4 illustrates both the vertical and horizontal displacement of an object.

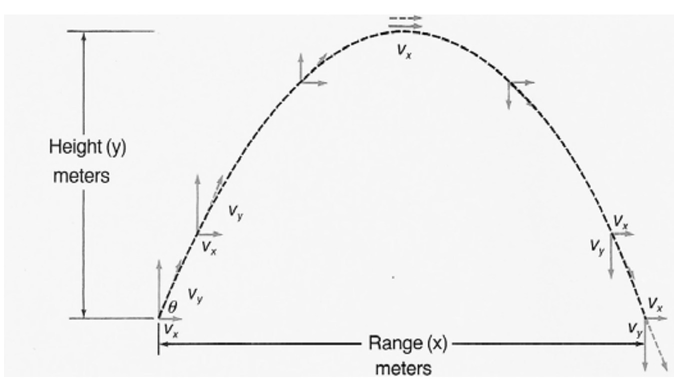


Figure 4[3]

Teaching The Mathematical Modeling Cycle

System/Reality

The system in this curriculum unit is projectile motion – the trajectory of an object moving through air. Students should consider examples of one- and two-dimensional motion. Driving a car and dropping a coin are examples of one-dimensional motion. Throwing, hitting, or kicking a ball into the air is an example of two-dimensional motion. From experience, they know that the ball will initially go up and then come back down.

Problem/questions

According to Joshua Abrams in his article *"Teaching Mathematical Modeling and the Skills of Representation,"*[4] the next step in the cycle is posing questions and identifying relevant variables. The relevant variables in this system are horizontal distance, vertical distance (especially maximum height), launch angle, launch velocity, initial height, and time in the air. After defining projectile motion and trajectory, I will ask students to brainstorm a list of factors that will affect the trajectory of an object launched into the air. I expect mass of the object to be included on the list, but experiments should convince them that neither vertical nor horizontal displacements depend on mass.

Armed with a list of variables, I will arrange students in groups of 3-4 students, and ask them to pose as many questions as possible about projectile motion. Next, I will ask them to categorize their questions according to whether they can be answered by collecting experimental data, answered by conducting research, mathematical modeling, etc. A sampling of questions might include

1. What is the effect of initial velocity on horizontal distance traveled?
2. How does the launch angle affect horizontal distance traveled?
3. What maximum height can be attained at different initial velocities?
4. How does initial velocity affect the time an object is in the air?
5. How does mass affect the time an object is in the air?
6. At what initial velocity does a rocket need to be launched vertically in order to reach a maximum height of 500 miles?
7. What horizontal distance can a model rocket launched at a 50° angle with an initial velocity of 200 ft/s travel?
8. What is the farthest a baseball was thrown?
9. What is the longest a football was in the air (hang time)?

Students will be able to answer the first five questions by performing experiments. Questions 6, 7 and 8 can be answered once mathematical models are developed, and the last two questions can most likely be answered by research. Within the group of questions that can be answered by experimentation, I will ask student to prioritize and/or order the questions in terms of what can and should be answered first, or prior to other questions. In other words, consider whether some questions depend on the results of others. Once the groups of students have compiled a list of categorized and prioritized questions, I will have them share them with the class. Using the SMART Board, the class can work with each category on a different page. Within the experimentation category, students can arrange and rank all questions posed by the class. At this point they can add more questions, and fit the new ones into the ranking list. The simplest questions, connecting only two variables, and easiest to measure, should be at the top of the list.

Formulate a mathematical model

To the extent possible, depending on the size of the class, each group will select a different question. Groups will need to design experiments and collect data in order to answer their question(s). I would expect them to vary the parameters as much as possible and repeat the experiment to ensure reproducibility.

During this phase of the modeling cycle, I expect students to collect and analyze data that will find relationships between 1) vertical displacement versus time, 2) horizontal displacement versus time, 3) horizontal displacement versus launch angle, 4) vertical displacement versus horizontal displacement, and 5) maximum height versus launch angle. I recognize that it will take several iterations of the cycle to find all of these relationships, but they need to observe these relationships to deepen their understanding of projectile motion.

To collect data relating vertical displacement versus time, in its simplest form, students can drop balls from different heights and measure the time it takes to reach the ground. Students should drop objects of varying mass from the same height in order to discover/verify that mass has no effect on vertical displacement. It will be more difficult to collect data for objects launched upward or horizontally (or any angle in between), rather than just dropped.

I have a set of adjustable, spring-loaded ball launchers made by PASCO for use in the classroom (with goggles!). The launchers have three initial velocity settings, and can be set to a launch angle between 0 - 90°, where 0° is horizontal and 90° is vertical. My science department also has electronic timers made by CPO Science. Photogates (producing a beam of light that starts and/or stops the timer when the beam is broken) can be plugged into the timers. These photogates can be used to measure initial velocity; the timer starts when the launched ball passes through the first light beam and stops when the ball passes through the second one. If the photogates are positioned right next to each other, the light beams are 2.5 cm apart. Initial velocity can be calculated as $v = \frac{\Delta x}{\Delta t} = \frac{2.5}{\Delta t}$.

In trial experiments, we found that initial velocity was very reproducible; the time varied ± 0.0001 seconds for a constant launch angle and velocity setting.

With the launchers, the simplest thing to measure is the horizontal distance traveled by the ball. In my test trials, after a single launch, I placed a piece of carbon paper under a second sheet of paper on the floor in the vicinity of the ball's landing site. The ball landed with enough force to make a mark on the paper, and it was quite simple to do repetitive trials. Students can also vary the launch angle and/or launch velocity and measure the horizontal distance and/or the time it takes a ball to reach the ground. However, they will need to recognize and account for the initial height of the launched ball or build a platform level with the launch height.

Accuracy will be seriously compromised if students depend on vision and reaction time to capture height and time data for vertical displacement. My plan is to teach them how to use *LoggerPro* computer software to gather data from videos. I describe the software program in a little more detail in Lesson 1, in the Classroom Activities section. In Lesson 3 I describe the use of simulations available on the Internet to collect additional data.

Once data is collected, students will look for trends and patterns in their data tables. They should also create scatterplots to help analyze their results. My students are all adept at using graphing calculators to display scatterplots. They should also be able to look at a scatterplot and predict what type of function (linear, polynomial/quadratic, exponential, etc.) would best fit the data. They can then use their calculators, or other technology, to find the best-fit regression line.

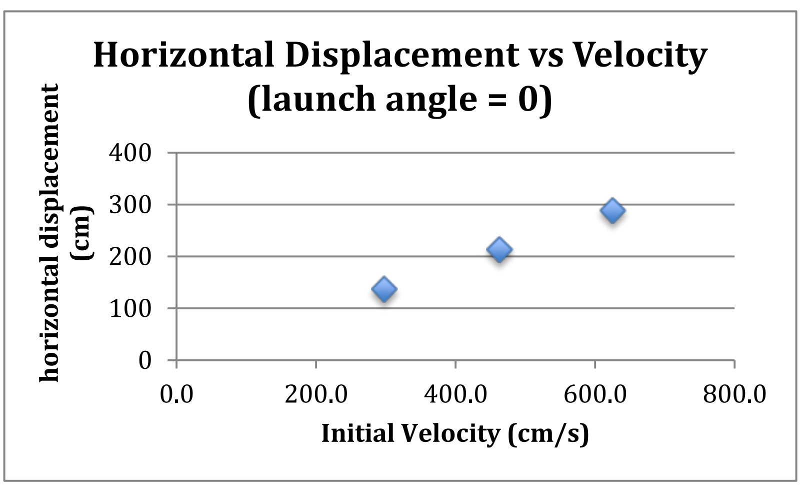


Figure 5

Figure 5 shows my experimental data using the PASCO launcher set on a table 106 cm above the ground. The ball was launched horizontally (angle = 0°) at three different velocities. The data appears to be linear, and a linear regression yields the equation $x = 0.46v - 1.1$, where x is the horizontal displacement and v is the initial velocity, with a correlation coefficient very nearly 1.

If students are like me, they will have more questions after their first round of data collection. These questions will lead to more experiments and more data collection, and more complete mathematical models, which is part of both the Modeling Cycle and the Scientific Method.

Analyze the model/simplification

In the analysis section, students should determine whether their models are sufficient to describe projectile motion. This stage will require class discussion. The first discussion will most likely confirm that mass does not affect vertical displacement; balls of different masses dropped (initial velocity is zero) from the same height reach the ground at the same time. Students should obtain a mathematical model close to $y = -4.9t^2 + h_0$, for vertical displacement, y, as a function of time, where h_0 is the initial drop height and the coefficient -4.9 is ½ the acceleration due to gravity in m/s^2.

The next discussion will be about horizontal displacement versus time. Students' mathematical models should be linear and have the form $x = vt$. This relationship should be straightforward to obtain for balls launched horizontally, as illustrated in Figure 5. Analysis of the data in Figure 5 shows that horizontal distance increased at a rate of 0.46 cm for every 1 cm/s increase in initial velocity. The y-intercept is not equal to zero in this example because horizontal displacement was not measured at the same height as the launched ball. Students may choose to build platforms to measure horizontal displacement at a constant height. That way, when the launch angle is increased, the data will show different horizontal distances for the same length of time in the air. At this point, I will be able to teach about vectors, specifically the horizontal and vertical components of the velocity vector. The mathematical model can then be written as $x = v_x \cdot t$, or $x = v \cdot \cos\theta \cdot t$.

Emphasizing that mathematical modeling is a cycle, future repetitions of the cycle will be able to include more inter-connected variables. Once students understand vector components, they should be able to use it to find the more complex relationships between vertical displacement and time, vertical displacement and horizontal displacement, and launch angle and maximum height. At this point, I will also work with them to derive *analytical* models (versus *empirical*) based on what they already know about physics/physical science and quadratic functions. More specifically, students know that vertical displacement (as in free fall) is dependent on gravity and is in the negative direction. From their math experience, they know that the initial height will be the y-intercept, or the constant in an equation. In a quadratic equation of the form $ax^2 + bx + c$, the linear term, bx, must be derived from upward velocity. Therefore, b can be replaced with $v_y = v \cdot sin\theta$, and the model becomes $y = -4.9t^2 + v \cdot sin\theta \cdot t + h_0$. Students with sufficient algebraic skills can use substitution to derive the mathematical model for vertical displacement as a function of horizontal displacement. Solving for t in $x = v \cdot cos\theta \cdot t$ yields $t = \frac{x}{v \cdot cos\theta}$. Substituting for t in the vertical displacement equation yields $y = \frac{-4.9}{v^2 \cdot cos^2\theta} \cdot x^2 + \frac{sin\theta}{cos\theta} \cdot x + h_0$, which is another quadratic equation. Students can test it with varying velocities and launch angles. Equations for maximum height and maximum horizontal distance can also be derived using algebraic and trigonometric substitutions based on $t = -\frac{b}{2a} = \frac{v \cdot sin\theta}{9.8}$, the time ($x$-coordinate) at the vertex of the parabola.

Prediction

The final stage in the modeling cycle is using a mathematical model to predict outcomes. Students will compare their experimental data to their models (function equations) to determine whether the models sufficiently describe the system. They can predict the answers to further questions such as “How long will it take for an object to reach the ground from a specific height?” or “What is the optimum angle and/or launch velocity to hit a target under given conditions?” They will need to perform more experiments to answer more questions and revise their mathematical models, adding to the complexity of it with each repetition of the cycle.

My ultimate goal is for students to have derived mathematical models for both vertical and horizontal displacement versus time, and recognize that the two relationships are independent. Finding a model for vertical displacement versus horizontal displacement can be done mathematically (analytically) and tested. The launch angle relationships are included in the displacement equations because initial velocity is broken into its components and used in the appropriate equation. For groups that finish more quickly, I will challenge them to find and test a mathematical model for vertical displacement as a function of launch angle in order to find the maximum height attainable.

Classroom Activities

The activities described in this section are all data collection activities. These activities will be part of the *Formulate a mathematical model* and *Analysis* steps of the Modeling Cycle. They follow the introductory and brainstorming discussions described in the *Problem/questions* step above.

Lesson 1: Dropping objects

Essential Question: How can free fall be described mathematically?

Activity

Students will bring in an assortment of (non-breakable) objects that they will drop from varying heights. This lesson has multiple objectives. It is an opportunity to introduce students to equipment that is available to collect data. The equipment may be as simple as meter sticks and stopwatches or electronic timers. I also want to introduce students to *Logger Pro* software by Vernier Software. It is available for Windows and Mac computers; it is also available as an App for iPads (probably other platforms, as well). *Logger Pro* allows students to take videos using the camera on a (laptop) computer or an uploaded video. In the video, students need to include something of a known length so they can scale the images. An important point to make to students is that whatever is being used to set the scale must be in the same plane (i.e. same distance from the camera) as the motion to get accurate results. For example, students could drop a ball next to the wall and use the height of a cinderblock, or the height of the blackboard to set the scale. Another objective is to address the Inquiry Science Standard. Students will design and organize their own experiments, making adjustments as needed, to answer questions about free fall. For each experiment, they should measure the mass of each object dropped, the initial height, and the time it takes to reach the ground. They will need to organize the data in tables and graphs using paper and pencil, or technology. Students will collect data in the classroom initially, but will be encouraged to move to stairwells to collect data from higher initial heights.

Assessment

The lab report for this activity should include the basic features – Purpose/Problem, Hypothesis, Apparatus, Procedure, Data, Evaluation of Data, Results/Discussion and Conclusion. Since one objective is to learn and practice inquiry, I will direct students to be explicit in their Procedure section. They should include their initial procedures and also explain adjustments they had to make as they worked through the experiment to get enough usable data to draw conclusions about the posed problem and prove/disprove their hypotheses.

The Evaluation of Data section of students' lab report coincides the *Formulate a mathematical model* step in the Modeling Cycle. Students will first create tables and graphs of their data. Then they will use what they have learned in math classes to find a mathematical model (equation for the line of best fit) for their data. I will also demonstrate how to use computer software, including *Logger Pro* and/or *Excel,* to get equations that fit the data. To demonstrate the *Prediction* step in the Modeling Cycle, students will use their models to predict the time for one last ball drop from a starting height, h_0. As stated earlier, students' models should be nearly $y = h(t) = -4.9t^2 + h_0$ (noticeably not including mass of the object!) for initial height measured in meters.

Lesson 2: Launching balls

Lesson 1 addressed one-dimensional motion in the vertical direction, and is essential before studying projectile motion in two dimensions. Students would have previous lab experience with linear motion in the horizontal direction when studying distance vs. displacement, speed vs. velocity, and acceleration. At some point during Lesson 2, I will need to teach students about vectors - drawing and performing operations on them – so they will be able to resolve the velocity vector into its horizontal and vertical components. This instruction will include a review of right triangle trigonometry to aid students in quantifying the effect of launch angle on vertical and horizontal velocities.

Essential Question: How can motion in two dimensions be described and quantified?

Activity

The first concept for students to understand in two-dimensional motion is the connection (actually, lack of connection) between horizontal velocity and the time it takes an object to reach the ground. I will have students walk and drop balls, and roll them off desks at different rates, each time measuring 1)initial height, 2) how long it takes for the ball to reach the ground and 3)where it first touches the ground (relative to where is was dropped). After analyzing their data, students should determine that the time for the ball to reach the ground is the same number of seconds they get from their mathematical models for free fall in Lesson 1. Substituting zero into their free fall models for *y*, they can calculate the time the ball is in the air, and see that it is independent of the horizontal motion before being dropped. It may sound simple here, but because there are three variables involved, students might need help in isolating the relationships of initial height versus time, and horizontal displacement versus time. Video recordings showing the ball's horizontal displacement after being dropped would probably be beneficial.

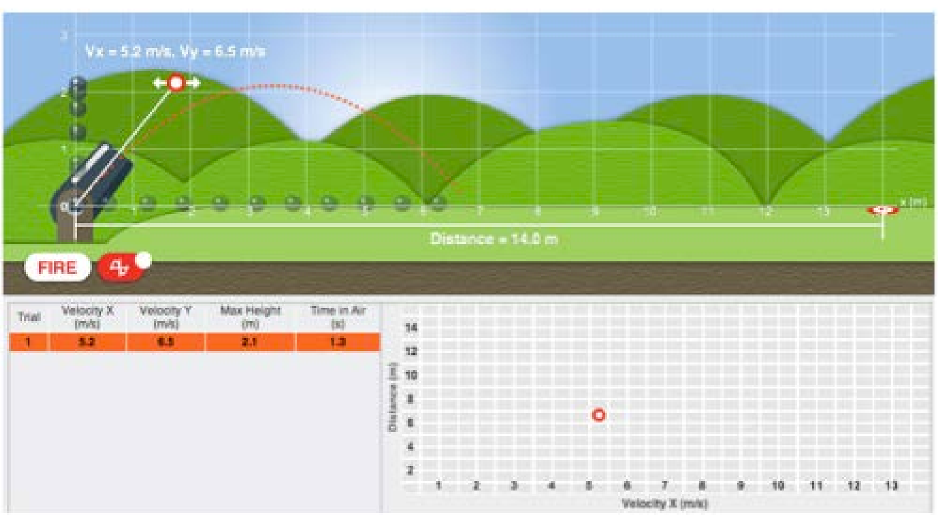
To build as complete a mathematical model as possible for projectile motion, students will have several days to design experiments and collect data using PASCO launchers, catapults, and even taking video cameras into gym classes to capture examples of projectile motion (e.g. basketball shots, soccer kicks, baseball/softball hits, etc.). The objective of their first few experiments is to address the questions recorded during the brainstorming session. If student groups are working on answering different questions, I will have each group present their results and mathematical models after one or two days, especially if these results are needed prior to addressing more complex questions from the list. As we learned in the Modeling Cycle, students should continue to raise more questions, adding to the original list, and add more parameters until they have a model detailed enough for their needs.

Assessment

Again, students' lab reports should describe the Inquiry process they went through in designing experiments to answer their questions. Their organized data tables and graphs should demonstrate that they have reproducible and reliable results for the effect of initial velocity and launch angle on vertical and horizontal displacement. As groups near the end of their experimentation, I will instruct them that their mathematical models should be able to predict the time it would take a ball to reach a specific height above the ground or specific horizontal position given the initial parameters of height and velocity.

Lesson 3: Simulations

Computer simulations of projectile motion will allow students to collect additional data in an ideal environment. Groups that struggled with data collection in Lesson 2 can use simulations to fill in gaps that they may need to complete their mathematical models. Groups can also test their models with the “ideal” data generated by simulations since the simulations are based on physical laws under ideal conditions. My plan is to use a cannon simulation that has free access on the Internet from CK-12. As seen in the screen shot in Figure 6, at Level 1 in the simulation, students can adjust the horizontal component of velocity; launch angle and vertical velocity are held constant. The ultimate goal of the simulation is to hit the target shown on the screen, but for our purposes, the data table and graph are just as useful. The screen also shows the constant horizontal velocity component and the changing vertical velocity component (decreasing in the upward direction to zero as the cannonball rises and increasing in the downward direction as it descends), which is reinforcement for the conceptual understanding of projectile motion. Higher levels are unlocked once a user hits the target. At Level 2, again, only horizontal velocity can be adjusted. Only vertical velocity can be adjusted at Level 3. Both velocity components can be adjusted at Level 4. The launch angle can be adjusted for a set launch velocity (although it’s labeled as speed on the screen) at Level 5. Once students hit the target at Level 5, they can move between levels to use the data, or create more data to build or improve their mathematical models. They can also use the data tables to practice resolving the velocity into its components (Level 5) or determining the launch angle from the given components (Levels 1 to 4).



Sample screen from Level 1 of CK-12 cannon simulation[5]
Figure 6

Assessment

The lab report for this lesson should include explanation and support of students' development of four mathematical models:

1. Horizontal displacement as a function of time
2. Vertical displacement as a function of time
3. Vertical displacement as a function of horizontal displacement
4. Vertical displacement as a function of launch angle

As a final assessment on projectile motion, my plan is to set up targets on the wall and the floor and have students use their mathematical models to determine where and how to position the launchers to hit the "bulls-eye." Because of the number of variables involved, students should even be able to find multiple set-up conditions to reach the targets. They can test their set-up(s) using computer simulation. Thus, the final assessment is also the final step in the Modeling Cycle: *Prediction*.

Materials

The minimal materials required to collect and analyze data for one- and two-dimensional motion are an assortment of balls, or objects of varying mass, a balance, a meter stick, timers/stopwatches, and graph paper. I would recommend graphing calculators and/or computers loaded with spreadsheet software, such as Microsoft Excel, that is capable of performing regressions to find the best-fitting line through a set of data points. Lastly, we need a way to launch balls to observe and measure two-dimensional motion. If there are no launchers or catapults available, students could build them as a project. An alternative is to take videos, or find appropriate simulations on the Internet. If schools have technology available – computers or iPads - I think the *LoggerPro* software/app would be a worthwhile investment.

Addressing District Standards

Delaware is adopting the Common Core State Standards for Mathematics (CCSS-M), so my District Math Standards follow CCSS.

<u>Standards for Mathematical Practice – MP4</u>: Model with mathematics

<u>High School - Modeling</u>: Modeling as a content standard is embedded throughout the other content standards

High School – Functions - HSF-IF.B.6: Interpret functions that arise in applications in terms of the context. Calculate and interpret the average rate of change of a function (presented symbolically or as a table) over a specified interval. Estimate the rate of change from a graph.

High School – Functions - HSF-LE.A.1b: Construct and compare linear, quadratic, and exponential models and solve problems. Recognize situations in which one quantity changes at a constant rate per unit interval relative to another.

High School – Geometry - G-SRT.C.8: Use trigonometric ratios and the Pythagorean Theorem to solve right triangles in applied problems.

District Science Standards follow the Delaware Science Standards. This unit addresses the following standards for grades 9-12:

Science Standard 1 – Nature and Application of Science and Technology: Science is a human endeavor involving knowledge learned through inquiring about the natural world. Scientific claims are evaluated and knowledge changes as a result of using the abilities and understandings of inquiry. The pursuit of scientific knowledge is a continuous process involving diverse people throughout history. The practice of science and the development of technology are critical pursuits of our society.

Science Standard 3 – Energy and Its Effects: The flow of energy drives processes of change in all biological, chemical, physical, and geological systems. Energy stored in a variety of sources can be transformed into other energy forms, which influence many facets of our daily lives. The forms of energy involved and the properties of the materials involved influence the nature of the energy transformations and the mechanisms by which energy is transferred. The conservation of energy is a law that can be used to analyze and build understandings of diverse physical and biological systems.

Strand IB: An object has kinetic energy because of its linear motion, rotational motion, or both. The kinetic energy of an object can be determined knowing its mass and speed. The object’s geometry also needs to be known to determine its rotational kinetic energy. An object can have potential energy when under the influence of gravity, elastic forces or electric forces and its potential energy can be determined from its position.

Resources

Abrams, Joshua Paul. "Teaching Mathematical Modeling and the Skills of Representation." In *The Roles of Representation in School Mathematics*. Reston, VA: National Council of Teachers of Mathematics, 2001. 269-282.
This is an alternate version of the Modeling Cycle with explanation and examples.

CK-12. "Cannon Sim." Apache Tomcat. http://simulations.ck12.org/CannonSim/ (accessed January 2, 2013).
This simulation is discussed in Lesson 3 above. The CK-12 website also has tutorials and question sets for a multitude of subjects, but its emphasis is math and science.

Common Core State Standards for Mathematics. http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf (accessed January 6, 2013).

State of Delaware Department of Education. http://www.doe.k12.de.us/infosuites/staff/ci/content_areas/science.shtml (accessed January 6, 2013).

Fendel, Daniel M., and Diane Resek. "High Dive-Circular Functions and the Physics of Falling Objects." In *Interactive mathematics program: integrated high school mathematics, year 3*. 2nd ed. Emeryville, Calif.: Key Curriculum Press, 2011. 417-491.
The Interactive Mathematics Program presents problems to students and leads them through activities to teach the required math to solve them. There are Teacher Resource books for each unit, also.

Fendel, Daniel M. "The Diver Returns-Circular Functions, Vector Components, and Complex Numbers." In *Interactive mathematics program: integrated high school mathematics, year 4*. 2nd ed. Emeryville, Calif.: Key Curriculum Press, 2012. 3-57.
This is a continuation of the previous textbook chapter that teaches vector components.

Hubbard, Miles J.. "Creating and Exploring Simple Models." *Mathematics Teacher* 101, no. 3 (2007): 193-199.
This article models projectile motion using the mathematical concept of sequences. It then creates the physical equations for velocity and acceleration by calculating the slope of a height versus time graph, and velocity versus time graph, respectively.

Lifelong Kindergarten Group at the MIT Media Lab. "Scratch | Home | imagine, program, share." Scratch | Home | imagine, program, share. http://scratch.mit.edu (accessed January 6, 2013).
Scratch is a free, downloadable programming language that could be used to teach basic programming skills. There are many simulation programs already available, which could be modified for specific needs.

"StarLogo TNG | MIT STEP." MIT STEP. http://education.mit.edu/projects/starlogo-tng (accessed January 6, 2013).
StarLogo is another free, downloadable programming language that could be used to

teach programming.

Zitzewitz, Paul, Mark Davids, Robert Neff, and Kelly Wedding. *Merrill physics: principles and problems*. New York: Glencoe, 1995.
This is one of many Physics textbooks available for high school students.

Curriculum Unit Title Mathematical Modeling of Motion **Author** Nancy Rudolph

KEY LEARNING, ENDURING UNDERSTANDING, ETC.

Scientific inquiry involves asking scientifically-oriented questions, collecting evidence, forming explanations, connecting explanations to scientific knowledge and theory, and communicating and justifying the explanation. Changes take place because of the transfer of energy. Energy is transferred to matter through the action of forces. Different forces are responsible for the transfer of the different forms of energy. Quantities and their relationships in physical, economic, public policy, social, and everyday situations can be modeled using mathematical and statistical methods. When making mathematical models, technology is valuable for varying assumptions, exploring consequences, and

ESSENTIAL QUESTION(S) for the UNIT

What makes a question scientific? What constitutes evidence? When do you know you have enough evidence? Why is it necessary to justify and communicate an explanation? How can energy be transferred from one material to another? What happens to a material when energy is transferred to it? What is the process of creating a mathematical model with quantitative and predictive value?

CONCEPT A	CONCEPT B	CONCEPT C
One-dimensional motion/Linear motion	Two-dimensional motion/Projecile motion	Mathematical modeling
ESSENTIAL QUESTIONS A	**ESSENTIAL QUESTIONS B**	**ESSENTIAL QUESTIONS C**
How can free fall be described mathematically?	How can motion in two dimensions be described and quantified?	How can mathematical models be used to predict set-up conditions for a projectile to hit a target?
VOCABULARY A	**VOCABULARY A**	**VOCABULARY A**
Speed (average and instantaneous), distance, displacement, velocity, acceleration, free fall, rate of change, force	Projectile motion, vector, component, resultant, resolution, scalar, parabolic, trajectory, quadratic function, vertex, initial height/velocity, angle of elevation	Analysis, predictive, regression, simulation, parameters

ADDITIONAL INFORMATION/MATERIAL/TEXT/FILM/RESOURCES

Required Resources: graphing capability (paper, calculator, computer); technology to perform Regression analyses, computers to access simulation software. Additional Resources: balls, balance, measuring devices (meter stick, tape), timers/stopwatches, launchers, catapults, video camera, *LoggerPro* computer software

Endnotes

[1] "Common Core State Standards for Mathematics," accessed January 6, 2013, http://www.corestandards.org/assets/CCSSI_Math%20Standards.pdf, 6.

[2] "Common Core State Standards for Mathematics."

[3] Zitzewitz, Paul et al., *Merrill physics: principles and problems*, 137.

[4] Abrams, Joshua Paul, "Teaching Mathematical Modeling and the Skills of Representation," in *The Roles of Representation in School Mathematics*, 272.

[5] "Cannon Sim," http://simulations.ck12.org/CannonSim/ accessed January 2, 2013.