Particle in a 1-Dimensional box

\[ \Psi = 0 \text{ outside the box} \]

The solution in the well:

\[ -\frac{\hbar^2}{2m} \frac{d^2 \Psi}{dx^2} = E \Psi \]

This must be solved subject to the condition that \( \Psi \) be continuous at all points in space

\[ \frac{d^2 \Psi}{dx^2} + \frac{2mE}{\hbar^2} \Psi = 0 \]

Particle in a 1-Dimensional box

\[ \frac{d^2 \Psi}{dx^2} + \frac{2mE}{\hbar^2} \Psi = 0 \]

Using trial method, can find that only exponential function works: \( \Psi(x) = \exp(sx) \)

\[ s^2 = -\frac{2mE}{\hbar^2} \text{ or } \]

\[ s = \pm i \frac{\sqrt{2mE}}{\hbar} = \pm i k \]

The general solution:

\[ \Psi(x) = A_+ \exp(ikx) + A_- \exp(-ikx) \]

or

\[ \Psi(x) = C \cos(kx) + D \sin(kx) \]
Particle in a 1-Dimensional box

\[ \Psi(x) = C \cos(kx) + D \sin(kx) \]

The singlevalued-ness criteria requires that the function is 0 at \( x = 0 \) and at \( x = a \).

The first result means that \( C = 0 \)

The second gives:

\[ \sin(ka) = 0, \text{ which is only true if } ka = n\pi, \]

where \( n \) is a positive integer.

Thus:

\[ i\sqrt{2mE \hbar} = ik = i\frac{\pi n}{a} \quad \text{and} \quad E_n = \frac{n^2 \hbar^2}{8ma^2} \quad \text{(here } \hbar = \frac{\hbar}{2\pi}) \]

The boundary conditions restrict energy to only have certain values. This is called quantization.

Constant \( D \) is not determined by previous conditions but can be found from normalization condition:

\[ \int_{-\infty}^{\infty} \psi^*(x)\psi(x)dx = \int_{0}^{a} \psi^*(x)\psi(x)dx = 1 \]

\[ \psi_n(x) = \frac{2}{\sqrt{a}} \sin \left( \frac{n\pi x}{a} \right) \]
Particle in a 1-Dimensional box

Particle in a 3-Dimensional box

Simple extension:

$0 \leq x \leq a; 0 \leq y \leq b; 0 \leq z \leq c$

$H = H_x + H_y + H_z$ such that

$H_x \psi_x = E_x \psi_x$

$H_y \psi_y = E_y \psi_y$

$H_z \psi_z = E_z \psi_z$

$E = E_x + E_y + E_z$

$\psi = \psi_x(x) \psi_y(y) \psi_z(z)$

$\psi(x,y,z) = \frac{8}{abc} \sin \left( \frac{n_x \pi x}{a} \right) \sin \left( \frac{n_y \pi y}{b} \right) \sin \left( \frac{n_z \pi z}{c} \right)$

$E = E_x + E_y + E_z = \frac{\hbar^2}{2m} \left[ \frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right]$
Particle in a 3-Dimensional box

The concept of degeneracy, \( g \):

Let’s make \( a = b = c \) (cubic box)

\[
E = E_x + E_y + E_z = \frac{\hbar^2}{8m} \left( \frac{n_x^2}{a^2} + \frac{n_y^2}{b^2} + \frac{n_z^2}{c^2} \right)
\]

but if \( a = b = c \), then:

\[
E = E_x + E_y + E_z = \frac{\hbar^2}{8ma^2} \left( n_x^2 + n_y^2 + n_z^2 \right)
\]

<table>
<thead>
<tr>
<th>(2,2,1)</th>
<th>(2,1,2)</th>
<th>(1,2,2)</th>
</tr>
</thead>
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<tr>
<td>(2,1,1)</td>
<td>(1,2,1)</td>
<td>(1,1,2)</td>
</tr>
<tr>
<td>(1,1,1)</td>
<td></td>
<td></td>
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</tbody>
</table>

Particle in a finite depth box
Comparison of finite and infinite potential

Finite potential and tunneling:
Scanning tunneling microscopy
Finite potential and tunneling: Scanning tunneling microscopy

Image of reconstruction on a clean Gold(100) surface

The 1-D Harmonic Oscillator
Quantum mechanical Harmonic Oscillator:
Model for Vibrational motion

The 1-D harmonic oscillator Hamiltonian

- Particle (mass m) attached to a spring of force constant, k
- Potential energy depends on position as a Hooke’s-law spring

\[ V = \frac{k}{2} (r - r_{eq})^2 = \frac{k}{2} x^2 \]

\[ H = \frac{p^2}{2m} + \frac{k}{2} x^2 \]
Classical solution of the 1-D harmonic oscillator

- Solve for trajectories for constant energy
  \[ x(t) = \sqrt{\frac{2E}{m\omega_0^2}} \cos\omega_0 t \]
  \[ p(t) = -\sqrt{2mE} \sin\omega_0 t \]
- Fundamental frequency, \( \omega_0 \)
- Oscillatory motion
- Maximum displacements are classical turning points
  \[ x_{\text{max}} = \pm \sqrt{\frac{2E}{m\omega_0^2}} \]

Quantum 1-D harmonic oscillator

- Schroedinger’s equation
  \[ H\Psi = E\Psi \]
  \[ -\frac{\hbar^2}{2m} \frac{\partial^2 \Psi}{\partial x^2} + \frac{k}{2} x^2 \Psi = E\Psi \]
- Convenient to make dimensionless equation
  \[ \frac{d^2 \Psi}{dy^2} - y^2 \Psi + \epsilon \Psi = 0 \]
  \[ y = \frac{x}{\alpha} \quad \alpha = \left( \frac{\hbar^2}{mk} \right)^{1/4} \quad \epsilon = \frac{2}{\hbar\omega_0} E \]
- Hermite’s associated differential equation
1-D harmonic-oscillator wave functions and energies

- Wavefunctions

\[ \Psi_{\nu}(x) = A_{\nu} H_{\nu}(x/\alpha) \exp \left( -\frac{x^2}{2\alpha^2} \right) \quad \nu = 0, 1, 2, 3, \ldots \]

- Energy eigenvalues

\[ E_{\nu} = \left( \nu + \frac{1}{2} \right) \hbar \omega_0 = \left( \nu + \frac{1}{2} \right) \hbar \nu_0 \]

Energy levels

- The 1-D harmonic oscillator has equally spaced energy states
- Energy spacing depends on the fundamental frequency
- Energy levels are nondegenerate
  - One state per level
Harmonic-oscillator wave functions

- Harmonic-oscillator wave functions are related to the Hermite polynomials
- Hermite polynomials are well-known sets of functions

<table>
<thead>
<tr>
<th>n</th>
<th>( H_n(y) )</th>
<th>Symmetry</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>1</td>
<td>Even</td>
</tr>
<tr>
<td>1</td>
<td>2y</td>
<td>Odd</td>
</tr>
<tr>
<td>2</td>
<td>4y^2 - 2</td>
<td>Even</td>
</tr>
<tr>
<td>3</td>
<td>8y^3 - 12y</td>
<td>Odd</td>
</tr>
</tbody>
</table>

Wave functions

- Hermite polynomials multiplied by a Gaussian function
- Note alternation in symmetry about \( x = 0 \)
  - Even
  - Odd
Probability Functions

- The square of the wave function gives the probability density at each position.
- Finite possibility the particle is outside of the classical turning points.

Quantum mechanical Harmonic Oscillator:
Model for Vibrational motion
Angular Momentum and the Rigid Rotor

Classical rigid rotor

- Rigid rotor model: A particle of mass m fixed to a massless rod
Classical rigid rotor

• Physical observables:
  angular velocity
  \[ |\omega| = \frac{d\theta}{dt} \]
  angular acceleration
  \[ \alpha = \frac{d|\omega|}{dt} = \frac{d^2\theta}{dt^2} \]
  kinetic energy
  \[ E_{\text{kin}} = \frac{1}{2} m v^2 = \frac{1}{2} I \omega^2 = \frac{1}{2} I \omega^2 \]

Angular momentum

• Vector property that describes circular motion of a particle or a system of particles
• Rigid rotor model: A particle of mass \( m \) fixed to a massless rod
• Examples
  – Swinging a bucket of water
  – Movement of the Earth around the Sun
Classical angular momentum

- Linear motion (Newton’s 2nd Law)
  \[ \mathbf{F} = m\mathbf{a} = \frac{d\mathbf{p}}{dt} \]
- Angular motion
  \[ \mathbf{L} = \mathbf{r} \times \mathbf{p}; \]
  \[ L = pr\sin \phi = \mu vr \sin \phi \]
  \[ E = \frac{p^2}{2\mu} = \frac{l^2}{2\mu r^2} = \frac{l^2}{2l} \]

Classical constant-angular-momentum problem

- Solve for trajectories for constant angular momentum
- Frequency, \( \omega \), must be constant
- \( r \) must be constant
- Constant \( \mathbf{L} \) is provided by the fact that \( r \) and \( \omega \) are constant

\[ \mathbf{L} = \text{constant} = mr^2\omega \mathbf{k} \]
\[ \mathbf{r}(t) = r(\mathbf{i} \cos \omega t + \mathbf{j} \sin \omega t) \]
\[ \mathbf{p}(t) = mr\omega(-\mathbf{i} \sin \omega t + \mathbf{j} \cos \omega t) \]
Quantum angular-momentum operators

• Vector definitions

\[
\mathbf{L} = L_x \mathbf{i} + L_y \mathbf{j} + L_z \mathbf{k}
\]

\[
L^2 = \mathbf{L} \cdot \mathbf{L} = L_x^2 + L_y^2 + L_z^2
\]

• Expression by correspondence

\[
\hat{L}_x = -i\hbar \left(y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}\right)
\]

\[
\hat{L}_y = -i\hbar \left(z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}\right)
\]

\[
\hat{L}_z = -i\hbar \left(x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x}\right)
\]

\[
\hat{L}^2 = \hat{L}_x^2 + \hat{L}_y^2 + \hat{L}_z^2
\]

• Form of operators with a fixed \( r \)

\[
\hat{\mathbf{L}} = -i\hbar \mathbf{r} \times \nabla
\]

\[
\hat{L}^2 = -\hbar^2 (\mathbf{r} \times \nabla) \cdot (\mathbf{r} \times \nabla)
\]

Quantum angular momentum

• Commutators of operators

\[
\left[\hat{L}_x, \hat{L}_y\right] = i\hbar \hat{L}_z \quad \text{and cyclic permutations}
\]

\[
\left[\hat{L}_x^2, \hat{L}_y\right] = 0
\]

• Can have common set of eigenstates of \( L^2 \) and any one component

\[
\hat{L}^2 \Psi_{km} = k\hbar^2 \Psi_{km}
\]

\[
\hat{L}_z \Psi_{km} = m\hbar \Psi_{km}
\]
Operators in spherical co-ordinates

• Natural system for describing angular motion is spherical co-ordinates

\[ \hat{L}_z = i\hbar \left( \sin \phi \frac{\partial}{\partial \theta} + \cot \theta \cos \phi \frac{\partial}{\partial \phi} \right) \]
\[ \hat{L}_r = -i\hbar \left( \cos \phi \frac{\partial}{\partial \theta} - \cot \theta \sin \phi \frac{\partial}{\partial \phi} \right) \]
\[ \hat{L}_\phi = -i\hbar \frac{\partial}{\partial \phi} \]

• \( L_z \) depends only on \( \phi \)
  – Suggests that the wave functions may be written as a product

\[ \Psi_{km}(\theta, \phi) = \Theta_{km}(\theta) \Phi_m(\phi) \]

Differential equations for angular-momentum eigenstates

• The \( z \) component yields a simple differential equation for \( \Phi_m \)
  \[ -i\hbar \frac{\partial \Phi_m}{\partial \phi} = \hbar \Phi_m \]

• The square of the angular momentum yields an equation for \( \Theta_{km} \) (\( \equiv P(\cos \theta) \))
  – Legendre's associated differential equation
  – Depend on a quantum number, \( \ell \)

\[ -\left( \frac{\partial^2 \Theta_{km}}{\partial \theta^2} + \cot \theta \frac{\partial \Theta_{km}}{\partial \theta} - \frac{m^2}{\sin^2 \theta} \Theta_{km} \right) = k \Theta_{km} \]

\[ Y_{\ell m}(\theta, \phi) = A_{\ell m} P_{\ell m}(\cos \theta) \Phi_m(\phi) \]

where
\[ k = \ell (\ell + 1) \quad \text{and} \quad \ell = 0, 1, 2, \ldots \]
Angular-momentum wave functions

- Functions of $\phi$ are exponentials
  \[ \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp(i m \phi) \]
- Legendre polynomials

| $l$ | $|m|$ | $P_{l}^{m}(\theta)$ |
|-----|------|---------------------|
| 0   | 0    | 1                   |
| 1   | 0    | $\cos \theta$      |
| 1   | 1    | $\sin \theta$      |
| 2   | 0    | $3 \cos^2 \theta - 1$ |
| 2   | 1    | $\sin \theta \cos \theta$ |
| 2   | 2    | $\sin^2 \theta$    |

- Should look familiar, as these are the angular parts of hydrogenic wave functions

Quantum rigid rotor

- Hamiltonian
  \[ \hat{H} = \frac{1}{2mr_0^2} \hat{L}^2 \]
- The Hamiltonian commutes with $L^2$ and $L_z$
  - The three operators have a complete set of eigenstates in common
  \[ \hat{H} Y_{lm}(\theta, \phi) = E_{lm} Y_{lm}(\theta, \phi) \]
  \[ \frac{1}{2mr_0^2} \hat{L}^2 Y_{lm}(\theta, \phi) = \frac{1}{2mr_0^2} \ell(\ell + 1) \hbar^2 Y_{lm}(\theta, \phi) \]
  \[ E_{lm} = \frac{\hbar^2}{2mr_0^2} \ell(\ell + 1) \]
Spherical harmonics

Grotrian diagram for the rigid rotor

- Rigid rotor’s energies determined by the quantum number, $\ell$

- Each energy level is degenerate
  - States with different values of $m$ have the same energy

$$g_\ell = 2\ell + 1$$
Spatial quantization of angular momentum

• $L_x$, $L_y$, $L_z$ cannot be known simultaneously – do not commute
• Can only know $|L|$ and one component, $L_z$
• $L=L_x+L_y+L_z$ cannot lie along $z$;

\[ L^2 - L_z^2 = L_x^2 + L_y^2 = \ell(\ell+1)\hbar^2 - m^2\hbar^2 \]

circle terminating the cone at its open end
• All the possible magnitudes of $L$ are quantized
• The vector can have only certain orientations in space

Spin

• Goudschmidt and Uehlenbeck proposed electronic “intrinsic angular momentum” to explain spectroscopic anomalies
• Fundamental property of particle called spin
  – Often labeled $I$ or $S$
  – Acts like other quantum angular momenta
  – Integer or half-integer values
• Dirac theory of an electron
  – Consequence of relativistic motion of electron

<table>
<thead>
<tr>
<th>PRINCIPAL SPIN QUANTUM NUMBERS OF PARTICLES</th>
</tr>
</thead>
<tbody>
<tr>
<td>Electron</td>
</tr>
<tr>
<td>Proton</td>
</tr>
<tr>
<td>Neutron</td>
</tr>
<tr>
<td>Deuteron</td>
</tr>
<tr>
<td>$^{12}$C</td>
</tr>
<tr>
<td>$^{13}$C</td>
</tr>
<tr>
<td>$^{23}$Na</td>
</tr>
<tr>
<td>$^{27}$Al</td>
</tr>
<tr>
<td>$^{63}$Cu and $^{65}$Cu</td>
</tr>
</tbody>
</table>
Hydrogen Atom

Hydrogen atom in quantum mechanics: electron moving about a proton located at the origin of the coordinate system

Coulomb potential

\[ U = -\frac{e^2}{4\pi \epsilon_0 |r|} = -\frac{e^2}{4\pi \epsilon_0 r} \]

Centrosymmetric potential, use spherical polar coordinates to formulate the Schrödinger equation:

\[
\begin{aligned}
\frac{-\hbar^2}{2m} \left[ &\frac{1}{r^2} \frac{\partial}{\partial r} \left( r^2 \frac{\partial \psi}{\partial r} \right) + \frac{1}{r \sin \theta} \frac{\partial}{\partial \theta} \left( \sin \theta \frac{\partial \psi}{\partial \theta} \right) + \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \psi}{\partial \phi^2} \right] \\
\end{aligned}
- \frac{e^2}{4\pi \epsilon_0 r} \psi(r, \theta, \phi) = E \psi(r, \theta, \phi)
\]

Hydrogen Atom: Solving the Schrödinger Equation

Separation of variables: since \( U(r) \) does not depend on the angles:

\[ \psi(r, \theta, \phi) = R(r) \Theta(\theta) \Phi(\phi) \]

Solution of the Schrödinger equation greatly simplified:

\[
\begin{aligned}
\frac{-\hbar^2}{2m} \left[ &\frac{d}{dr} \left( r^2 \frac{dR}{dr} \right) + \frac{1}{2m} R(r)^2 \left[ \frac{\hbar^2 (l^2 + 1)}{2m} - \frac{2e^2}{4\pi \epsilon_0 r^2} \right] \right] \psi(r, \theta, \phi) = E \psi(r, \theta, \phi) \\
\end{aligned}
\]

Effective potential, centrifugal + Coulomb
Hydrogen Atom: Eigenvalues and Eigenfunctions of Total Energy

Energy appears only in the radial equation (not angular):

\[ E_n = -\frac{m_e e^4}{8\varepsilon_0^2 \hbar^2 n^2}, \quad n = 1, 2, 3, 4, \ldots \]

Bohr radius:

\[ a_0 = -\frac{\varepsilon_0^2 \hbar^2}{\pi m_e e^2}, \quad a_0 = 0.529 \times 10^{-10} \text{ m} \]

Energy taking Bohr radius into account:

\[ E_n = -\frac{e^4}{8\pi\varepsilon_0 a_0 n^2} = -\frac{2.179 \times 10^{-18} J}{n^2} = -\frac{13.60 eV}{n^2}, \quad n = 1, 2, 3, 4, \ldots \]

Hydrogen Atom: Atomic Emission Spectra

Experimental frequencies of H atom emission spectra:

\[ \nu = \frac{1}{\hbar} \left| E_{\text{initial}} - E_{\text{final}} \right| \]

\[ E_n = -\frac{m_e e^4}{8\varepsilon_0^2 \hbar^2 n^2}, \quad n = 1, 2, 3, 4, \ldots \]

\[ \nu = \frac{\mu e^4}{8\varepsilon_0^2 \hbar^2} \left( \frac{1}{n_{\text{initial}}^2} - \frac{1}{n_{\text{final}}^2} \right) \]

\[ \mu = \frac{m_e m_p}{m_e + m_p} \quad \tilde{\nu} = \frac{\nu}{c} = \frac{1}{\lambda} \quad \frac{m_e e^4}{8\varepsilon_0^2 \hbar^2 c} = 109,677.581 \text{ cm}^{-1} \]

Reduced mass Wave number Rydberg constant
Hydrogen Atom: Atomic Emission Spectra

Lyman series (n'=1, UV band)
Balmer series (n'=2, visible band)
Paschen series (n' = 3, IR band)
Brackett series (n' = 4)
Pfund series (n' = 5)
Humphreys series (n' = 6)

Hydrogen Atom: Eigenvalues and Eigenfunctions of Total Energy

Eigenfunctions:

\[ \psi_{n,l,m_l}(r, \theta, \phi) = R_{nl}(r) Y_{l,m_l}^m(\theta, \phi) \]

Quantum numbers:

\[ n = 1, 2, 3, 4, \ldots \]
\[ l = 0, 1, 2, 3, \ldots, n-1 \]
\[ m_l = 0, \pm 1, \pm 2, \pm 3, \ldots, \pm l \]
Hydrogen Atom: Orbitals

**Eigenfunctions:**

\[ n = 1, \ l = 0, \ m_r = 0 \quad \psi_{100}(r) = \frac{1}{\sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right) e^{-r/a_0} \]

\[ n = 2, \ l = 0, \ m_r = 0 \quad \psi_{200}(r) = \frac{1}{4 \sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right)^2 e^{-r/a_0} \]

\[ n = 2, \ l = 1, \ m_r = 0 \quad \psi_{210}(r, \theta, \phi) = \frac{1}{4 \sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right)^2 \frac{\sin \theta \cos \phi}{a_0} \]

\[ n = 2, \ l = 1, \ m_r = \pm 1 \quad \psi_{21m}(r, \theta, \phi) = \frac{1}{8 \sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right)^2 e^{-r/a_0} \sin \theta \sin \phi \]

\[ n = 3, \ l = 2, \ m_r = 0 \quad \psi_{320}(r, \theta, \phi) = \frac{1}{8 \sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right)^2 e^{-r/a_0} (3 \cos^2 \theta - 1) \]

Degeneracy in energy levels: \( n^2 \)

---

Hydrogen Atom: Orbitals

To visualize orbitals- normalize the eigenfunctions and make linear combinations

\[ N^2 \int_0^\infty \psi^*(z) \psi(z) \, dz = 1; \quad N^2 \int_0^\pi \sin \theta \, d\theta \int_0^{2\pi} \, d\phi \int_0^a \psi_{100}(r, \theta, \phi) \psi_{100}(r, \theta, \phi) = 1 \]

\[ \psi_{100}(r, \theta, \phi) = \frac{1}{4 \sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right) e^{-r/a_0} \sin \theta \cos \phi \]

\[ \psi_{200}(r, \theta, \phi) = \frac{1}{4 \sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right)^2 e^{-r/a_0} \sin \theta \sin \phi \]

\[ \psi_{210}(r, \theta, \phi) = \frac{1}{4 \sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right)^2 \frac{\sin \theta \cos \phi}{a_0} \]

\[ \psi_{211}(r, \theta, \phi) = \frac{1}{8 \sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right)^2 e^{-r/a_0} \sin \theta \sin \phi \]

\[ \psi_{21m}(r, \theta, \phi) = \frac{1}{8 \sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right)^2 e^{-r/a_0} \sin \theta \cos \phi \]

\[ \psi_{321}(r, \theta, \phi) = \frac{1}{8 \sqrt{\pi}} \frac{1}{a_0} \left( \frac{r}{a_0} \right)^2 e^{-r/a_0} (3 \cos^2 \theta - 1) \]

---

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Hydrogen Atom: Orbitals

3D orbital contour plots

Radial wave function vs. r

Hydrogen Atom: Orbitals

2D orbital contour plots:

\[ R(r) - n - l - l \] nodal surfaces

\[ Y_l^m(\theta, \phi) - l \] nodal surfaces

\[ R(r) Y_l^m(\theta, \phi) - n - l \] nodes

(same as in particle in the box and harmonic oscillator)
Hydrogen Atom: Radial Distribution Function

What is the probability of finding an electron at a particular value of $r$ regardless of $\theta$ and $\phi$?

Integrate probability density

$$\psi_{nlm}^2 (r, \theta, \phi) \ r^2 \sin \theta \ dr \ d\theta$$

over all values of $\theta$ and $\phi$

Radial distribution function:

$$P_{nl} (r) dr = \int_0^\infty \int_0^{2\pi} \left( \psi_{nlm}^2 (r, \theta, \phi) \right) r^2 \sin \theta d\theta d\phi \ r^2 R_{nl}^2 (r) dr$$

Hydrogen Atom: The Validity of the Shell Model

- Broad maxima rather than sharp boundaries
- Additional nodes and subsidiary maxima - manifestation of wave behavior (standing waves and interference)