Angular momentum

- Vector property that describes circular motion of a particle or a system of particles
- Rigid rotor model: A particle of mass m fixed to a massless rod
- Examples
  - Swinging a bucket of water
  - Movement of the Earth around the Sun
  - \( L = 2.5 \times 10^{40} \) kg m² s⁻¹

Classical constant-angular-momentum problem

- Solve for trajectories for constant angular momentum
- Frequency, \( \omega \), must be constant
- \( r \) must be constant
- Constant \( L \) is provided by the fact that \( r \) and \( \omega \) are constant

\[
L = \text{constant} = mr^2 \omega \hat{k}
\]

\[
r(t) = r(\cos \omega t + j \sin \omega t)
\]

\[
p(t) = m\omega r(-i \sin \omega t + j \cos \omega t)
\]

Quantum angular momentum

- Commutators of operators
  \[
  [\hat{L}_z, \hat{L}_x] = i\hbar \hat{L}_y \quad \text{and cyclic permutations}
  \]
  \[
  [\hat{F}, \hat{L}_z] = 0
  \]

- Can have common set of eigenstates of \( \hat{L}^2 \) and any one component

\[
\hat{L}_z^2 \Psi_{km} = k \hbar^2 \Psi_{km}
\]

\[
\hat{L}_z \Psi_{km} = m \hbar \Psi_{km}
\]

Quantum angular-momentum operators

- Vector definitions

\[
\hat{L} = \hat{L}_x + \hat{L}_y + \hat{L}_z
\]

\[
\hat{L}^2 = \hat{L}_x \hat{L}_y + \hat{L}_y \hat{L}_z + \hat{L}_z \hat{L}_x
\]

- Expression by correspondence

\[
\hat{L}_z = -i \hbar (\frac{\partial}{\partial \phi} - \frac{\cot \theta}{\sin \theta} \frac{\partial}{\partial \theta})
\]

\[
\hat{L}_x = -i \hbar \frac{\partial}{\partial \theta}
\]

- Form of operators with a fixed \( r \)

\[
\hat{L} = -i \hbar \hat{r} \times \nabla
\]

\[
\hat{L}^2 = -\hbar^2 (\hat{r} \times \nabla) \cdot (\hat{r} \times \nabla)
\]

Operators in spherical coordinates

- Natural system for describing angular motion is spherical coordinates
- \( \hat{L}_z \) depends only on \( \phi \)
- Suggests that the wave functions may be written as a product

\[
\Psi_{km}(\theta, \phi) = \Theta_{km}(\theta) \Phi_{m}(\phi)
\]
Differential equations for angular-momentum eigenstates

- The $z$ component yields a simple differential equation for $\Phi_m$:
  \[ -i\hbar \frac{\partial \Phi_m}{\partial \phi} = m\hbar \Phi_m \]

- The square of the angular momentum yields an equation for $\Theta_{km} (= P(k\cos \theta))$:
  \[ \frac{\partial^2 \Theta_{km}}{\partial \phi^2} + \frac{\partial^2 \Theta_{km}}{\partial \theta^2} + \frac{\sin \theta}{\theta} \frac{\partial \Theta_{km}}{\partial \theta} \left( \frac{1}{\sin \theta \cos \theta} - \frac{1}{\theta} \right) \Theta_{km} - k^2 \Theta_{km} = 0 \]

- Legendre's associated differential equation depends on a quantum number, $\lambda$.
- Solutions are a complete set called the spherical harmonic functions.

Angular-momentum wave functions

- Functions of $\phi$ are exponentials:
  \[ \Phi_m(\phi) = \frac{1}{\sqrt{2\pi}} \exp(im\phi) \]

- Legendre polynomials

<table>
<thead>
<tr>
<th>$k$</th>
<th>$\lambda$</th>
<th>$P_k^\lambda(\cos \phi)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>$1$ (constant)</td>
</tr>
<tr>
<td>1</td>
<td>1</td>
<td>$\cos \phi$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$3\cos^2 \phi - 1$</td>
</tr>
<tr>
<td>2</td>
<td>2</td>
<td>$3\cos^2 \phi + 1$</td>
</tr>
</tbody>
</table>

- Should look familiar, as these are the angular parts of the hydrogenic wave functions.

Quantum rigid rotor

- Hamiltonian:
  \[ \hat{H} = \frac{1}{2m^2} \hat{L}^2 \]

- The Hamiltonian commutes with $L^2$ and $L_z$.
- The three operators have a complete set of eigenstates in common:
  \[ \hat{H}_{km}(\theta, \phi) = E_{km}(\theta, \phi) \]
  \[ \frac{1}{2m^2} \hat{E} Y_{km}(\theta, \phi) = \frac{1}{2m^2} \ell(\ell+1) Y_{km}(\theta, \phi) \]
  \[ E_{km} = \frac{\hbar^2}{2m^2} \ell(\ell+1) \]

Grotrian diagram for the rigid rotor

- Rigid rotor's energies determined by the quantum number, $\ell$.
- Each energy level is degenerate:
  - States with different values of $m$ have the same energy:
    \[ g_\ell = 2\ell + 1 \]

Spin

- Goudscheim and Uehlenbeck proposed electronic “intrinsic angular momentum” to explain spectroscopic anomalies.
- Fundamental property of particle called spin:
  - Often labeled $I$ or $S$.
  - Acts like other quantum angular momenta.
  - Integer or half-integer values.
- Dirac theory of an electron:
  - Consequence of relativistic motion of electron.

Summary

- Angular momentum is quantized:
  - Combination of:
    - Rotation equation
    - Legendre’s differential equation
    - Restricted values of $\ell$ and $m$:
      - $|m|$ must be less than or equal to $\ell$
      - $m$ must be an integer
  - Rigid rotor:
    - Hamiltonian is directly proportional to $L^2$.
    - Same set of eigenstates.
    - Degenerate levels:
      - $g_\ell = 2\ell + 1$