Chapter 13
Partial Differential Equations

1 Scope of the Chapter

This chapter is concerned with the numerical solution of partial differential equations. At present the only module provided is for the three dimensional Helmholtz equation. It should be noted, however, that this includes the Laplace and Poisson equations as special cases.

Modules for more general steady and time-dependent problems are planned at future releases of NAG *fl*90.

2 Available Modules

Module 13.1: `nag_pde_helm` — **Helmholtz equations**

Provides a procedure for solving the Helmholtz equation in three dimensions.

3 Background

3.1 Helmholtz Equation

The elliptic equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} + \lambda u = f(x, y, z), \tag{1}$$

where λ is a real constant, is generally referred to as *Helmholtz equation*, or in the case $\lambda > 0$ as the *reduced wave equation*. The problem is usually posed in a region V bounded by a closed surface S. Typical boundary conditions include:

Dirichlet: solution u given on S;

Neumann: derivative $\dfrac{\partial u}{\partial n}$ given on S, where n is the outward unit normal to S.

In practice Dirichlet conditions will often be prescribed on part of the boundary and Neumann conditions on the remainder. Periodic boundary conditions are also common for certain simple regions D.

For the case $\lambda < 0$ a straightforward application of the Green's identity

$$\int_V (u\nabla^2 u + \mathrm{grad}^2 u)\mathrm{d}V = \int_S u\frac{\partial u}{\partial n}\mathrm{d}S$$

can be used to prove uniqueness of the solution of (1) with any combination of Dirichlet and Neumann boundary conditions in a bounded region V. For $\lambda = 0$ the solution can be proved to be unique provided a Dirichlet condition is given for at least one point on S; the solution of the *pure* Neumann problem is unique only to within an arbitrary additive constant.

For the reduced wave equation ($\lambda > 0$) there are no corresponding uniqueness theorems for bounded regions. The equation commonly arises in the theory of acoustic and electromagnetic scattering, when solutions of the form $u(x, y, z)e^{i\omega t}$ are sought for the wave equation

$$c^2\left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}\right) - \frac{\partial^2 \phi}{\partial t^2} = 0. \tag{2}$$

This results in equation (1), where λ is given by $(\omega/c)^2$.