Chapter c02 – Zeros of Polynomials

1. Scope of the Chapter

This chapter is concerned with computing the zeros of a polynomial with real or complex coefficients.

2. Background

Let $f(z)$ be a polynomial of degree n with complex coefficients a_i:

$$f(z) \equiv a_0 z^n + a_1 z^{n-1} + a_2 z^{n-2} + \ \dots \ + a_{n-1} z + a_n, \qquad a_0 \neq 0.$$

A complex number z_1 is called a *zero* of $f(z)$ (or equivalently a *root* of the *equation* $f(z) = 0$), if:

$$f(z_1) = 0.$$

If z_1 is a zero, then $f(z)$ can be divided by a factor $(z - z_1)$:

$$f(z) = (z - z_1) f_1(z)$$

where $f_1(z)$ is a polynomial of degree $n-1$. By the Fundamental Theorem of Algebra, a polynomial $f(z)$ always has a zero, and so the process of dividing out factors $(z - z_i)$ can be continued until we have a complete *factorization* of $f(z)$

$$f(z) \equiv a_0 (z - z_1)(z - z_2) \ \dots \ (z - z_n).$$

Here the complex numbers $z_1, z_2, \dots, z_n$ are the zeros of $f(z)$; they may not all be distinct, so it is sometimes more convenient to write:

$$f(z) \equiv a_0 (z - z_1)^{m_1} (z - z_2)^{m_2} \ \dots \ (z - z_k)^{m_k}, \qquad k \leq n,$$

with distinct zeros $z_1, z_2, \dots, z_k$ and multiplicities $m_i \geq 1$. If $m_i = 1$, z_i is called a *single* zero, if $m_i > 1$, z_i is called a *multiple* or *repeated* zero; a multiple zero is also a zero of the derivative of $f(z)$.

If the coefficients of $f(z)$ are all real, then the zeros of $f(z)$ are either real or else occur as pairs of complex conjugate numbers $x + iy$ and $x - iy$. A pair of complex conjugate zeros are the zeros of a quadratic factor of $f(z)$, $(z^2 + rz + s)$, with real coefficients r and s.

A zero is called *ill-conditioned* if it is sensitive to small changes in the coefficients of $f(z)$. Conversely, a zero is called *well-conditioned* if it is comparatively insensitive to such perturbations. Roughly speaking, a zero which is well separated from other zeros is well–conditioned, while zeros which are close together are ill–conditioned, but in talking about 'closeness' the decisive factor is not the absolute distance between neighbouring zeros but their *ratio*: if the ratio is close to 1 the zeros are ill–conditioned. In particular, multiple zeros are ill–conditioned. A multiple zero is usually split into a cluster of zeros by perturbations in the polynomial or computational inaccuracies.

The accuracy of the roots will depend on how ill-conditioned they are. Peters and Wilkinson (1971) describe techniques for estimating the errors in the zeros after they have been computed.

3. Reference

Peters G and Wilkinson J H (1971) Practical Problems Arising in the Solution of Polynomial Equations *J. Inst. Math. Appl.* **8** 16–35.

4. Available Functions

Zeros of a complex polynomial	`c02afc`
Zeros of a real polynomial	`c02agc`