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Chapter 1

Boussinesq Models and Applications to Nearshore Wave Propagation, Surf Zone Processes and Wave-Induced Currents

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1. INTRODUCTION

Classical Boussinesq theory provides a set of evolution equations for surface water waves in the combined limit of weak nonlinearity (characterized by $\delta \ll 1$) and weak dispersion ($\mu \ll 1$) with the ratio $\delta/\mu^2 = O(1)$. The parameters represent a wave height to water depth ratio, and a water depth to wavelength ratio, respectively. In an early review of the state of modeling efforts directed at predicting wave-induced nearshore circulation, Basco (1983, p. 352–353) concluded that

> "The Boussinesq theory offers the possibility to eventually raise the fundamental knowledge of coastal hydrodynamics to a higher level. No time-averaging is involved. Nonlinear wave propagation and resulting wave height variations are automatically produced as part of the calculation procedure. The unsteady asymmetrical currents and instantaneous water surface variations as solutions to the governing equations are only obtainable with the aid of large, high-speed computers. Solution techniques and applications are in their infancy. Wave breaking and surf zone simulations have yet to be implemented."

At the time of this prediction, Boussinesq models were scarce, difficult and time consuming to run, and relatively undeveloped for practical physical applications. Very few explicit calculations of coastal wave propagation, and none of surf zone processes, had been made using models based on the Boussinesq theory, and the long-term averaging of model results needed to obtain predictions of mean currents had not been performed. The conclusion that the Boussinesq model approach could provide an advantage over the more well-established procedure of using a radiation stress field to drive a slowly varying mean current field (for a recent example see Özkan-Haller and Kirby, 1999) was met by occasional skepticism, as evidenced for example by the discussion of Basco's paper by Kirby and Dalrymple (1984).

In the years from 1983 to the present, events have firmly indicated that Basco was correct in his original assessment. Modeling schemes based on Boussinesq equations coupled with innovative extensions to the theoretical framework have been shown to be accurate and revealing predictors of a wide range of nearshore hydrodynamic behavior, including wave propagation and shoaling, wave current interaction, wave breaking and the generation of nearshore circulation, wave-structure interaction and a range of additional topics. The availability of faster computers is bringing the modeling technique into the realm of practical calculations, and model codes have been documented and are, in some cases (for example, Kirby et al., 1998) freely available to the public.

This chapter provides an overview of several aspects of the recent development of the Boussinesq modeling technique, aimed especially at providing a description or predictive capability in the nearshore ocean. The review highlights the work of this author and collaborating colleagues at the University of Delaware over the past decade, and is thus in some sense somewhat narrow in its orientation. Attempts have been made to provide balanced indications of the work of other groups in the field, but for omissions apologies are offered.

The chapter proceeds by providing an overview of the development of modern fully nonlinear Boussinesq theory in Section 2, and provides several examples illustrating wave shoaling and propagation properties as well as a test of the generation and advection of a vertical-core vortex structure. Section 3 turns to practical extensions to the $O(\mu^2)$ theory, which provides the foundation of most operational Boussinesq models at this time. Extensions to curvilinear coordinates and the inclusion of wave breaking, bottom friction and subgrid-scale mixing are described and illustrated. Section 4 discusses the problem of nearshore circulation, and describes example calculations addressing the generation of longshore currents, longshore current instability and formation of shear waves, and the generation and destabilization of rip currents. Section 5 discusses recent results including vertical shear effects (or the presence of horizontal vorticity). Miscellaneous topics are discussed in Section 6, including the use of Boussinesq models to assist in the depth inversion problem, and an application to tsunami propagation and inundation.

2. BOUSSINESQ EQUATIONS FOR WAVE PROPAGATION

The onset of recent developments in the field of Boussinesq models was triggered by two events. The first was the increasing availability of the computer resources needed to run the models. The second was the development of variants of the theory which could be optimized to obtain better dispersion properties at larger kh values, thus allowing the model to treat a larger range of water depths. Critical steps in this process were provided by Madsen et al. (1991), who established a procedure for optimizing model performance through rearrangement of dispersive terms, and Nwogu (1993), who demonstrated the flexibility obtained by using the horizontal velocity at a given elevation in the water column as a dependent variable. Both procedures have been extensively utilized in the development of subsequent theory. The review articles of Kirby (1997) and Madsen and Schäffer (1999) and the book by Dingemans (1997) provide extensive reviews of these developments up to 1999, and therefore the material here is concentrated on more recent developments. In addition, the subjects of wave interaction with permeable structures, surf and swash zone sediment transport, and frequency domain modeling are covered elsewhere in this book and are thus largely neglected here.

2.1. Hydrodynamic Fundamentals

Almost all Boussinesq-type models are derived from the framework of incompressible, inviscid flow. (The exception will be the case of waves with horizontal vorticity or vertical shear, considered in Section 5) To proceed, a scaling which is appropriate to the regime where wavelength exceeds water depth is chosen.

$$(x, y) = (k_0 x', k_0 y'); z = z'/h_0; t = \sqrt{g h_0 k_0^2} t'; \eta = \eta'/a_0; \phi = \left(\frac{a_0\sqrt{g h_0}}{k_0 h_0}\right)^{-1} \phi' \tag{1}$$

where primes denote dimensional variables, and where h_0 is a depth scale, a_0 is a wave amplitude scale, and k_0 is an inverse horizontal length scale. The dependent variables are surface displacement η and velocity potential ϕ. Velocity components are then given by

$$\mathbf{u} = (u, v) = \nabla\phi \tag{2}$$

for horizontal velocities, where $\nabla = (\partial/\partial x, \partial/\partial y)$, and

$$w = \phi_z \tag{3}$$

for vertical velocity, where subscripts x, y, z or t will denote partial derivatives. The resulting scaled problem is characterized by the dimensionless ratios

$$\mu = k_0 h_0; \qquad \delta = a_0 / h_0 \tag{4}$$

The parameter μ characterizes frequency dispersion, and the limit $\mu \to 0$ represents the non-dispersive limit. The designation *weakly dispersive* refers to the regime $\mu \ll 1$. The parameter δ characterizes nonlinearity, and the limit $\delta \to 0$ represents the linear limit. The designation *weakly nonlinear* refers to the regime $\delta \ll 1$. In the present context, we will use the terminology *fully non-linear* to indicate that no truncation based on powers of δ is employed in obtaining the corresponding model equations. The resulting set of scaled equations is given by

$$\nabla^2 \phi + \frac{1}{\mu^2} \phi_{zz} = 0, \qquad -h \leqslant z \leqslant \delta\eta \tag{5}$$

$$\nabla h \cdot \nabla \phi + \frac{1}{\mu^2} \phi_z = 0, \qquad z = -h \tag{6}$$

$$\eta + \phi_t + \frac{\delta}{2} \left(|\nabla \phi|^2 + \frac{1}{\mu^2} (\phi_z)^2 \right) = 0, \qquad z = \delta\eta \tag{7}$$

$$\eta_t + \delta \nabla \eta \cdot \nabla \phi - \frac{1}{\mu^2} \phi_z = 0, \qquad z = \delta\eta \tag{8}$$

Equation (8) is often replaced by a depth-integrated form of equation (5) which uses equations (6) and (8) to resolve boundary terms, giving

$$\eta_t + \nabla \cdot \mathbf{M} = 0; \qquad \mathbf{M} = \int_{-h}^{\delta\eta} \nabla \phi dz \tag{9}$$

The central feature of Boussinesq wave theories is that the solution to equations (5)–(6) is usually given as a power series in z, after which the surface boundary conditions are employed to obtain evolution equations. The choice of a reference elevation for the series expansion in z is initially fairly arbitrary. Following Madsen and Schäffer (1998), an expansion about the still water level of the form

$$\phi(x, y, z, t) = \sum_{n=0}^{\infty} z^n \phi^{(n)}(x, y, t) \tag{10}$$

gives, after substitution in equation (5),

$$\phi(x, y, z, t) = \sum_{n=0}^{\infty} (-1)^n \mu^{2n} \left(\frac{z^{2n}}{(2n)!} \nabla^{2n} \phi^{(0)} + \frac{z^{2n+1}}{(2n+1)!} \nabla^{2n} \phi^{(1)} \right) \tag{11}$$

where $\phi^{(0)}$ and $\phi^{(1)}$ represent ϕ and ϕ_z evaluated at $z = 0$ and are unknown prior to applying boundary conditions. Agnon et al. (1999) demonstrate that equation (11) recovers the full linear solution for a slowly varying plane wave over a mild bottom slope.

The standard procedure for developing Boussinesq models follows from using the bottom boundary condition (equation (6)) to eliminate the $\phi^{(1)}$ unknown in favor of $\phi^{(0)}$ (or a suitably defined replacement), after which the development proceeds using a truncated expansion for ϕ. We concentrate on this procedure below. More recently, Agnon et al. (1999) and Madsen et al. (2002) have pursued a path where the identity of horizontal and vertical velocities is maintained through much of the derivation, adding a dependent variable but increasing flexibility in optimizing results. This procedure will be discussed in Section 2.4.

2.2. The Weakly Dispersive Problem

The full linear problem taken from equations (5)–(8) describes propagating water waves with a dispersion relation given by

$$\omega^2 = gk \tanh kh \tag{12}$$

or, equivalently,

$$c^2 = gh \frac{\tanh kh}{kh} \tag{13}$$

where ω denotes angular frequency, k denotes magnitude of a wave number vector $\mathbf{k}$, and c denotes phase speed for a monochromatic wave component. In the limit $\mu \ll 1$, the ratio in equation (13) should approach 1, and hence $c^2 \sim gh$. The resulting waves are nearly non-dispersive, with a leading order correction of the form

$$c^2 = gh(1 + O(kh)^2) \tag{14}$$

Obviously, an approximation of this form can impose severe restrictions when the problem of propagating water waves in a general domain with a range of water depths is considered. In modern terms, the principal goal of most derivations of Boussinesq models is to obtain an approximation to the ratio in equation (13) which is fairly robust over a range of values of kh; i.e., extending outside of the limit $\mu \to 0$.

Following Nwogu (1993), we define a reference elevation z_α located within the water column, and re-express the series expansion for ϕ in terms of the value at z_α. Using the bottom boundary condition (equation (6)) and truncating the resulting series after $O(\mu^2)$ gives

$$\phi(x, y, z, t) = \phi_\alpha(x, y, t) + \mu^2(z_\alpha - z)\nabla \cdot (h\nabla\phi_\alpha) - \frac{1}{2}\mu^2(z_\alpha^2 - z^2)\nabla^2\phi_\alpha + O(\mu^4) \tag{15}$$

Substituting equation (15) in linearized versions of equations (7) and (9) (with $\delta = 0$) and using $\phi_\alpha \sim \exp i(kx - \omega t)$ gives the dispersion relation

$$\omega^2 = gk^2h \frac{1 - (\alpha + 1/3)(kh)^2}{1 - \alpha(kh)^2} \tag{16}$$

where

$$\alpha = \frac{1}{2}\left(\frac{z_\alpha}{h}\right)^2 + \frac{z_\alpha}{h} \tag{17}$$

The choice of α fixes the resulting dispersion relation and the corresponding value of z_α. $\alpha = -1/3$ reproduces the classical Boussinesq theory based on depth-averaged velocity, while the choice $\alpha = -2/5$ reproduces the $(2, 2)$ Padé approximant to equation (13). Nwogu further adjusted the second result by choosing α to minimize a measure of phase speed error over a range of kh values, and obtained $\alpha = -0.39$. A comparison of the true linear dispersion relation (equation (13)) and the approximate form (equation (16)) is shown in Figure 1, and shows that reasonably accurate dispersion can be obtained for a range of μ values up to about 3. The next order of approximation is discussed in Section 2.4.

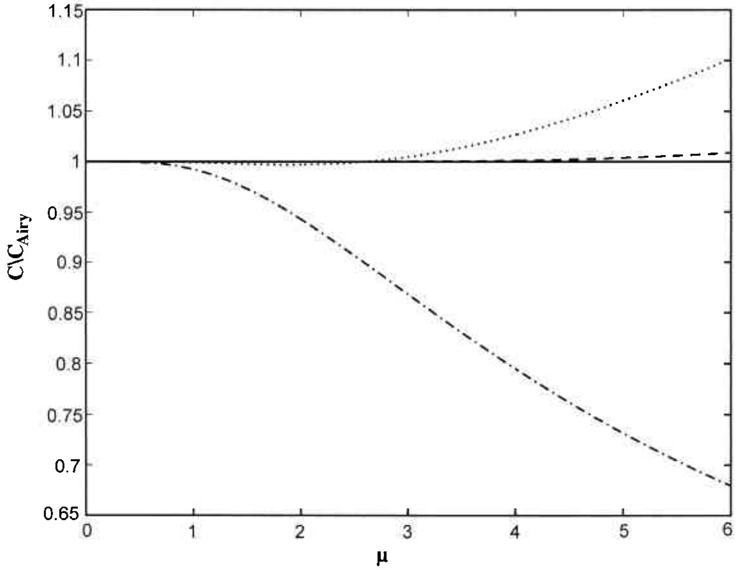

Figure 1. Ratio of model linear phase speed c to exact linear phase speed c_{Airy} given by equation (13). Standard Boussinesq dispersion with $\alpha = -1/3$ in equation (16) (dash-dot); optimized $O(\mu^2)$ dispersion based on the (2,2) Padé approximant with minimization of phase speed errors to obtain $\alpha = -0.39$ in (16), following Nwogu (1993) (dotted); $O(\mu^4)$ dispersion based on (4,4) Padé approximant in (31) (dash). (From Gobbi et al., 2000).

2.3. Weak vs. Full Nonlinearity in the $O(\mu^2)$ Boussinesq Formulation

Subsequent to the initial work on improved linear dispersion, the next topic to receive attention was the problem of relaxing the restriction of weak nonlinearity in the problem formulation. The need for this extension is clear when one realizes that the wave height to water depth ratio essentially is of $O(1)$ in the surf zone and just seaward of it. The most obvious line of approach staying within the Boussinesq type of formulation is to drop the notion of pursuing an expansion in powers of δ, and instead use the weakly dispersive expression for ϕ or horizontal velocity $\mathbf{u}$ in the form of a power series in μ^2 to evaluate the complete surface boundary condition. We will subsequently refer to models resulting from this procedure as *fully nonlinear models*, in the sense that all of the available information on velocities is used to evaluate the full boundary conditions. Numerous early examples of this approach appear in the literature and have been reviewed by Dingemans (1997), Kirby (1997) and Madsen and Schäffer (1999). We restrict our attention here to two examples; those of Wei et al. (1995) and Liu (1994). Following the procedure of Nwogu (1993), each study derived a set of model equations for potential flow written initially in terms of ϕ_α and η given by the volume conservation equation (9) and the Bernoulli equation (7). Using expression (15) in equations (9) and (7) gives

$$\mathbf{M} = H\left[\nabla\phi_\alpha + \mu^2\left\{\nabla\left[z_\alpha\nabla\cdot(h\nabla\phi_\alpha) + \frac{z_\alpha^2}{2}\nabla^2\phi_\alpha\right]\right.\right.$$
$$\left.\left. + \frac{(h-\delta\eta)}{2}\nabla(\nabla\cdot(h\nabla\phi_\alpha)) - \frac{(h^2 - h\delta\eta + (\delta\eta)^2)}{6}\nabla^2\nabla\phi_\alpha\right\}\right] \tag{18}$$

for volume flux (where $H = h + \delta\eta$ is the total water depth), and

$$\eta + \phi_{\alpha t} + \frac{\delta}{2}\nabla\phi_\alpha \cdot \nabla\phi_\alpha + \mu^2\left[(z_\alpha - \delta\eta)\nabla\cdot(h\nabla\phi_{\alpha t}) + \frac{1}{2}(z_\alpha^2 - (\delta\eta)^2)\nabla^2\phi_{\alpha t}\right]$$
$$+ \delta\mu^2\left\{\nabla\phi_\alpha \cdot \left[\nabla z_\alpha \nabla\cdot(h\nabla\phi_\alpha) + (z_\alpha - \delta\eta)\nabla(\nabla\cdot(h\nabla\phi_\alpha))\right]\right\}$$
$$+ \delta\mu^2\left\{\nabla\phi_\alpha \cdot \left[z_\alpha \nabla z_\alpha \nabla^2\phi_\alpha + \frac{1}{2}(z_\alpha^2 - (\delta\eta)^2)\nabla(\nabla^2\phi_\alpha)\right]\right\}$$
$$+ \delta\mu^2\left\{\frac{1}{2}[\nabla\cdot(h\nabla\phi_\alpha)]^2 + \delta\eta\nabla\cdot(h\nabla\phi_\alpha)\nabla^2\phi_\alpha + \frac{1}{2}(\delta\eta)^2(\nabla^2\phi_\alpha)^2\right\} = 0 \tag{19}$$

for the Bernoulli equation. These results were found independently by Liu (1994) and Wei et al. (1995).

An alternate model in terms of η and horizontal velocity $\mathbf{u}_\alpha$ at the reference level z_α is preferred for practical use, as it is extendable to include breaking, frictional and mixing effects. Substituting for $\nabla\phi_\alpha$ in equation (18) using

$$\nabla\phi_\alpha = \mathbf{u}_\alpha - \mu^2[\nabla z_\alpha \nabla\cdot(h\mathbf{u}_\alpha) + z_\alpha \nabla z_\alpha \nabla\cdot\mathbf{u}_\alpha] + O(\mu^4) \tag{20}$$

gives

$$\mathbf{M} = H\left[\mathbf{u}_\alpha + \mu^2\left\{\left[\frac{1}{2}z_\alpha^2 - \frac{1}{6}(h^2 - h\delta\eta + (\delta\eta)^2)\right]\nabla(\nabla\cdot\mathbf{u}_\alpha)\right.\right.$$
$$\left.\left. + \left[z_\alpha + \frac{1}{2}(h - \delta\eta)\right]\nabla(\nabla\cdot(h\mathbf{u}_\alpha))\right\}\right] + O(\mu^4) \tag{21}$$

for volume flux. Taking the horizontal gradient of equation (19) leads to a horizontal momentum equation which may be written schematically in the form

$$\mathbf{u}_{\alpha t} + \delta(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + \nabla\eta + \mu^2\mathbf{V_1} + \delta\mu^2\mathbf{V_2} = O(\mu^4) \tag{22}$$

In deriving their version of the model equation, Wei et al. (1995) erroneously made the substitution

$$\frac{\delta}{2}\nabla(\mathbf{u}_\alpha \cdot \mathbf{u}_\alpha) \to \delta(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha \tag{23}$$

after using equation (20) in the Bernoulli equation. The substitution (equation (23)) implies the incorporation of a vorticity term $\boldsymbol{\omega} \times \mathbf{u}_\alpha$, but, as will be shown below, the $O(\mu^2)$ contribution to $\boldsymbol{\omega}$ is missed in this substitution, as pointed out by Chen et al. (2000b) and Hsaio et al. (2002). Wei et al. (1995) obtained the dispersive terms

$$\mathbf{V_1} = \frac{1}{2}z_\alpha^2\nabla(\nabla\cdot\mathbf{u}_{\alpha t}) + z_\alpha\nabla(\nabla\cdot(h\mathbf{u}_{\alpha t})) - \nabla\left[\frac{1}{2}(\delta\eta)^2\nabla\cdot\mathbf{u}_{\alpha t} + \delta\eta\nabla\cdot(h\mathbf{u}_{\alpha t})\right] \tag{24}$$

$$\mathbf{V_2} = \mathbf{V_{2W}} = \nabla\left[(z_\alpha - \delta\eta)(\mathbf{u}_\alpha\cdot\nabla)(\nabla\cdot(h\mathbf{u}_\alpha)) + \frac{1}{2}(z_\alpha^2 - (\delta\eta)^2)(\mathbf{u}_\alpha\cdot\nabla)(\nabla\cdot\mathbf{u}_\alpha)\right]$$
$$+ \frac{1}{2}\nabla\left[(\nabla\cdot(h\mathbf{u}_\alpha) + \delta\eta\nabla\cdot\mathbf{u}_\alpha)^2\right] \tag{25}$$

where the $\mathbf{W}$ subscript in $\mathbf{V_{2W}}$ denotes the Wei et al. version. In contrast, Liu (1994) (see corrected versions in Lynett et al., 2002) invoked the substitution

$$\frac{\delta}{2}\nabla(\nabla\phi_\alpha\cdot\nabla\phi_\alpha) = \delta(\nabla\phi_\alpha\cdot\nabla)\nabla\phi_\alpha \tag{26}$$

within the original gradient of the Bernoulli equation, and obtained the expression $\mathbf{V_1}$ as in equation (24). After some rearrangement to get a form close to the Wei et al. form $\mathbf{V_{2W}}$, which is a pure gradient, Liu's $\mathbf{V_2}$ may be written as

$$\mathbf{V_{2L}} = \mathbf{V_{2W}} + \mathbf{V_{2r}} \tag{27}$$

where

$$\begin{aligned}\mathbf{V_{2r}} = {} & (\mathbf{u}_\alpha \cdot \nabla z_\alpha)\left[z_\alpha \nabla(\nabla \cdot \mathbf{u}_\alpha) + \nabla(\nabla \cdot (h\mathbf{u}_\alpha))\right] \\ & -\nabla z_\alpha (\mathbf{u}_\alpha \cdot \nabla)(\nabla \cdot (h\mathbf{u}_\alpha)) - z_\alpha \nabla z_\alpha (\mathbf{u}_\alpha \cdot \nabla)(\nabla \cdot \mathbf{u}_\alpha)\end{aligned} \tag{28}$$

The physical interpretation of this extra term was discovered by Chen et al. (2000b). After some manipulation, $\mathbf{V_{2r}}$ may be written in the compact form

$$\mathbf{V_{2r}} = \boldsymbol{\omega}_\mathbf{1} \times \mathbf{u}_\alpha \tag{29}$$

where $\boldsymbol{\omega}_\mathbf{1}$ denotes the $O(\mu^2)$ contribution to vertical vorticity, given by

$$\begin{aligned}\boldsymbol{\omega}_\mathbf{1} &= \mathbf{i_z}\left[z_{\alpha,x}\left[(\nabla \cdot (h\mathbf{u}_\alpha))_{,y} + z_\alpha(\nabla \cdot \mathbf{u}_\alpha)_{,y}\right] - z_{\alpha,y}\left[(\nabla \cdot (h\mathbf{u}_\alpha))_{,x} + z_\alpha(\nabla \cdot \mathbf{u}_\alpha)_{,x}\right]\right] \\ &= \nabla z_\alpha \times \nabla[\nabla \cdot (h\mathbf{u}_\alpha) + z_\alpha \nabla \cdot \mathbf{u}_\alpha]\end{aligned} \tag{30}$$

Adding $\mathbf{V_{2r}}$ to equation (25) corrects the original model of Wei et al. (1995). We will refer to the combined set of terms simply as $\mathbf{V_2}$ subsequently. Note that the correction term does not appear in weakly-nonlinear formulations, where terms of $O(\delta\mu^2)$, and hence all of $\mathbf{V_2}$, are neglected. The term also vanishes in water of constant depth, where z_α is constant.

Wei and Kirby (1995) have described a numerical scheme for equations of this type which has come into fairly wide usage. Time stepping is treated using a fourth-order Adams-Bashforth-Moulton scheme, while spatial differencing is handled using a mixed-order scheme, employing fourth-order accurate centered differences for first derivatives and second-order accurate derivatives for third derivatives. The latter choice is made in order to move leading truncation errors to one order higher than the $O(\mu^2)$ dispersive terms, while maintaining the tridiagonal structure of spatial derivatives within time-derivative terms. Wei and Kirby (1995) used a non-staggered grid scheme with $\mathbf{u}_\alpha$ and η defined at the same locations. More recently, Shi et al. (2001a) have used a staggered grid approach which has less apparent sensitivity to treatment of boundary conditions. The staggered grid scheme has become our preferred approach. Methods for generating waves at internal sources have been described by Wei et al. (1999) and Chawla and Kirby (2000). Kirby et al. (1998) document a version of the non-staggered code, known as *FUNWAVE*, which is available at http://chinacat.coastal.udel.edu/~kirby/programs/funwave.

Properties of this model for wave propagation problems have been reviewed by Kirby (1997) and Madsen and Schäffer (1999). The ability of the model to provide an accurate representation of the evolution and transport of the vertical vorticity component was tested by Hommel et al. (2000), who compared model results to laboratory data for the case of a vertical vortex core shed during the passage of a solitary wave past a vertical plane wall blocking half the width of a wave flume. These tests were partially motivated by a previous study by Roddier and Ertekin (1999), who had considered the diffraction of a solitary wave at the tip of a breakwater using a potential flow model analogous to the Bernoulli equation formulation (equation (19)). Roddier and Ertekin indicated the formation of a deep depression at the breakwater tip, which they explained to be a "bathtub vortex". However, a consideration of their geometry shows that a vortex core could not be forming at a position attached to the tip of the wall, since the presence of the wall would interrupt the circulation of fluid around the

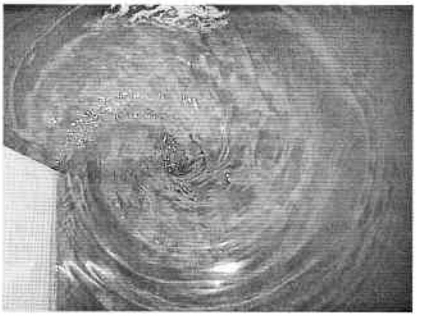

Figure 2. Shed vortex formed during passage of solitary wave past a vertical wall. (From Hommel et al., 2000. Reproduced with permission of ASCE).

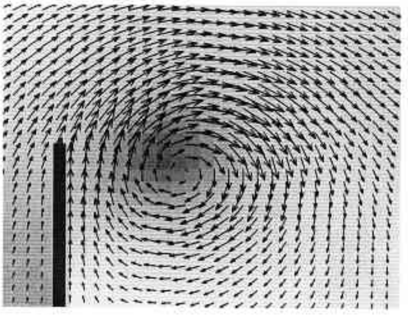

Figure 3. Calculated velocity field corresponding to vortex in Figure 2. (From Hommel et al., 2000. Reproduced with permission of ASCE).

depression. The depression is simply the manifestation of a singularity in the solution at the breakwater tip, caused by the approach to infinite acceleration as fluid turns the 180° corner. This result shows conclusively that the truly irrotational models in the form of a Bernoulli equation will not spontaneously generate a vortex due to flow separation or advect the vortex away from the generation region. Hommel et al. (2000) considered the somewhat different case of a breakwater oriented parallel to the crest of the approaching solitary wave. Figure 2 shows a photograph of a vortex core shed during passage of a solitary wave from the left, with the core rotation in the clockwise direction. Figure 3 shows the corresponding numerical result, with velocity vectors overlying a colormap of the vorticity field. In general, agreement between modeled and measured velocity time series were good for these cases. Results are omitted here, and the reader is referred to Hommel et al. (2000).

2.4. Extensions to Higher Order

The $O(\mu^2)$ model described above has provided a robust framework for predicting wave propagation in shallow to intermediate water depths, but still exhibits limitations in predicting water particle kinematics. In particular, predictions of wave-induced horizontal velocity near the sea bed breaks down for values of μ far smaller than implied by the limitations on model dispersion accuracy. For example, the horizontal velocity predicted by the model of Wei et al. (1995) falls to zero at $\mu = \sqrt{10}$ and becomes negative in sign (relative to the surface velocity) in deeper water, as illustrated in Figure 4. This type of result renders the $O(\mu^2)$ models useless for prediction of near bed kinematics

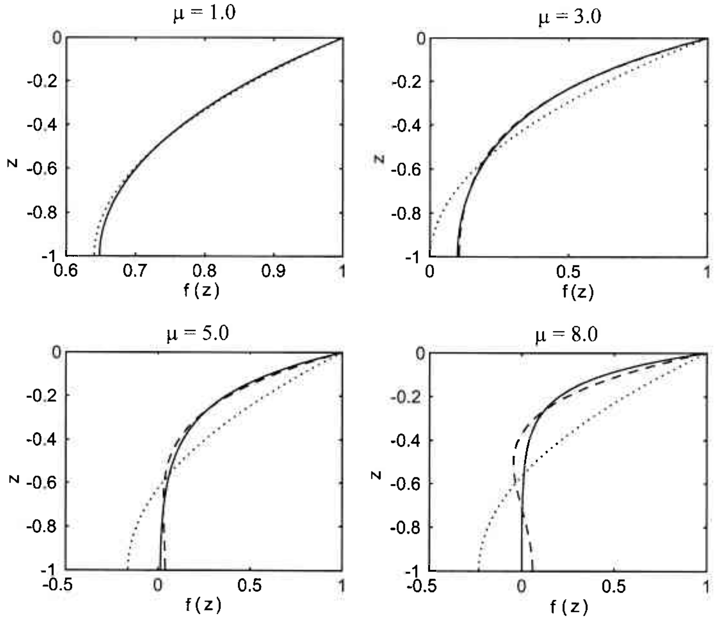

Figure 4. Normalized vertical profile of linear horizontal velocity at four values of μ. Exact linear solutions (solid), $O(\mu^2)$ approximate solution (dot), $O(\mu^4)$ approximate solution (dash) (From Gobbi et al., 2000).

(as would be needed in sediment transport calculations) at water depths far smaller than implied by the accuracy of the dispersion relation, unless an artificial means is employed to estimate near bed velocities from surface values. As a result, the development of higher-order approximations in the Boussinesq model has proceeded with the intent both of increasing the range of allowed water depths and with improving the accuracy of kinematic predictions within the allowed range of depths. Two avenues of approach are reviewed here.

2.4.1. Model of Gobbi et al. (2000)

One avenue of approach is to retain higher-order terms in the expansion for ϕ, and then proceed in constructing either the two-equation model in η, ϕ^* or the three-equation model in $\eta, \mathbf{u}^*$, where superscript $*$ indicates a particular choice of reference level or combination of reference levels used to specify the value of the potential or horizontal velocity. This procedure poses difficulties right away, as the method proposed by Nwogu (1993) does not provide access to higher-order versions of the Padé form of the dispersion relation (Dingemans, 1997). For example, at $O(\mu^4)$ in the expansion for ϕ, the desired expansion of the usual dispersion relation is given by

$$\frac{\tanh \mu}{\mu} = \frac{1 + (1/9)\mu^2 + (1/945)\mu^4}{1 + (4/9)\mu^2 + (1/63)\mu^4} + O(\mu^{10}) \tag{31}$$

The resulting phase speed estimate is shown in Figure 1. Gobbi et al. (2000) approached this problem by constructing a potential formulated as the weighted average of the potential at two Nwogu-type reference levels

$$\tilde{\phi} = \beta\phi_a + (1 - \beta)\phi_b \tag{32}$$

where ϕ_a and ϕ_b are defined as in the previous section and are evaluated at levels z_a and z_b. Relationships between these parameters giving the appropriate dispersion relationship are given by (Gobbi et al., 2000)

$$z_a = \left[\frac{1}{9} - \left\{ \frac{8\beta}{567(1-\beta)} \right\}^{1/2} + \left\{ \frac{8}{567\beta(1-\beta)} \right\}^{1/2} \right]^{1/2} - 1 \tag{33}$$

$$z_b = \left[\frac{1}{9} - \left\{ \frac{8\beta}{567(1-\beta)} \right\}^{1/2} \right]^{1/2} - 1 \tag{34}$$

Values $0.018 < \beta < 0.467$ give z_a and z_b levels lying within the water column and recover the form (equation (31)). The resulting truncated potential is given by

$$\phi = \tilde{\phi} + \mu^2 \left[(Ah - \zeta) F_1(\tilde{\phi}) + \left(Bh^2 - \zeta^2 \right) F_2(\tilde{\phi}) \right] + \mu^4 \left[(Ah - \zeta) F_3(\tilde{\phi}) \right.$$
$$\left. + \left(Bh^2 - \zeta^2 \right) F_4(\tilde{\phi}) + \left(Ch^3 - \zeta^3 \right) F_5(\tilde{\phi}) + \left(Dh^4 - \zeta^4 \right) F_6(\tilde{\phi}) \right] \tag{35}$$

where $\zeta = h + z$, and where

$$A \equiv \frac{1}{h} \left[\beta(h + z_a) + (1 - \beta)(h + z_b) \right] \tag{36}$$

$$B \equiv \frac{1}{h^2} \left[\beta(h + z_a)^2 + (1 - \beta)(h + z_b)^2 \right] \tag{37}$$

$$C \equiv \frac{1}{h^3} \left[\beta(h + z_a)^3 + (1 - \beta)(h + z_b)^3 \right] \tag{38}$$

$$D \equiv \frac{1}{h^4} \left[\beta(h + z_a)^4 + (1 - \beta)(h + z_b)^4 \right] \tag{39}$$

and

$$F_1(\tilde{\phi}) \equiv G \nabla h \cdot \nabla \tilde{\phi} \tag{40}$$

$$F_2(\tilde{\phi}) \equiv \frac{1}{2} G \nabla^2 \tilde{\phi} \tag{41}$$

$$F_3(\tilde{\phi}) \equiv \nabla h \cdot \nabla \left(Ah \nabla h \cdot \nabla \tilde{\phi} \right) + \frac{1}{2} \nabla h \cdot \nabla \left(Bh^2 \nabla^2 \tilde{\phi} \right) \tag{42}$$

$$F_4(\tilde{\phi}) \equiv \frac{1}{2} \nabla^2 \left(Ah \nabla h \cdot \nabla \tilde{\phi} \right) + \frac{1}{4} \nabla^2 \left(Bh^2 \nabla^2 \tilde{\phi} \right)$$
$$- \frac{1}{2} \nabla^2 h \nabla h \cdot \nabla \tilde{\phi} - \nabla h \cdot \nabla \left(\nabla h \cdot \nabla \tilde{\phi} \right) \tag{43}$$

$$F_5(\tilde{\phi}) \equiv -\frac{1}{6} \nabla^2 h \nabla^2 \tilde{\phi} - \frac{1}{3} \nabla h \cdot \nabla \left(\nabla^2 \tilde{\phi} \right) - \frac{1}{6} \nabla^2 \left(\nabla h \cdot \nabla \tilde{\phi} \right) \tag{44}$$

$$F_6(\tilde{\phi}) \equiv -\frac{1}{24} \nabla^2 \left(\nabla^2 \tilde{\phi} \right) \tag{45}$$

where $G = (1 + \mu^2 |\nabla h|^2)^{-1}$. Seeking a model system in terms of a horizontal velocity, Gobbi and Kirby (1999) introduced the definition

$$\tilde{\mathbf{u}}(x, y, t) = \beta \left[\nabla \phi \right]_{z=z_a} + (1 - \beta) \left[\nabla \phi \right]_{z=z_b} \tag{46}$$

where the relationship between $\tilde{\mathbf{u}}$ and $\tilde{\phi}$ is given by

$$\nabla\tilde{\phi} = \tilde{\mathbf{u}} - \mu^2 \nabla h \left[(A-1)\,F_{21} + 2\,(B-A)\,h F_{22}\right]$$
$$- \mu^4 \nabla h \left[(A-1)\,(F_{41} + F_{43}) + 2\,(B-A)\,h\,(F_{42} + F_{44})\right.$$
$$\left. + \; 3\,(C-B)\,h^2 F_{45} + 4\,(D-C)\,h^3 F_{46}\right] \tag{47}$$

where

$$F_{21}(\tilde{\mathbf{u}}) \equiv G\nabla h \cdot \tilde{\mathbf{u}} \tag{48}$$

$$F_{22}(\tilde{\mathbf{u}}) \equiv \frac{1}{2} G \nabla \cdot \tilde{\mathbf{u}} \tag{49}$$

$$F_{41}(\tilde{\mathbf{u}}) \equiv -|\nabla h|^2 \left[(A-1)\,\nabla h \cdot \tilde{\mathbf{u}} + (B-A)\,h \nabla \cdot \tilde{\mathbf{u}}\right] \tag{50}$$

$$F_{42}(\tilde{\mathbf{u}}) \equiv -\frac{1}{2} \nabla \cdot \left\{\nabla h \left[(A-1)\,\nabla h \cdot \tilde{\mathbf{u}} + (B-A)\,h \nabla \cdot \tilde{\mathbf{u}}\right]\right\} \tag{51}$$

$$F_{43}(\tilde{\mathbf{u}}) \equiv \nabla h \cdot \nabla \left(A h \nabla h \cdot \tilde{\mathbf{u}}\right) + \frac{1}{2} \nabla h \cdot \nabla \left(B h^2 \nabla \cdot \tilde{\mathbf{u}}\right) \tag{52}$$

$$F_{44}(\tilde{\mathbf{u}}) \equiv \frac{1}{2} \nabla^2 \left(A h \nabla h \cdot \tilde{\mathbf{u}}\right) + \frac{1}{4} \nabla^2 \left(B h^2 \nabla \cdot \tilde{\mathbf{u}}\right)$$
$$- \frac{1}{2} \nabla^2 h \nabla h \cdot \tilde{\mathbf{u}} - \nabla h \cdot \nabla \left(\nabla h \cdot \tilde{\mathbf{u}}\right) \tag{53}$$

$$F_{45}(\tilde{\mathbf{u}}) \equiv -\frac{1}{6} \nabla^2 h \nabla \cdot \tilde{\mathbf{u}} - \frac{1}{3} \nabla h \cdot \nabla \left(\nabla \cdot \tilde{\mathbf{u}}\right) - \frac{1}{6} \nabla^2 \left(\nabla h \cdot \tilde{\mathbf{u}}\right) \tag{54}$$

$$F_{46}(\tilde{\mathbf{u}}) \equiv -\frac{1}{24} \nabla^2 \left(\nabla \cdot \tilde{\mathbf{u}}\right) \tag{55}$$

The volume flux $\mathbf{M}$ in equation (9) is then given by

$$\mathbf{M} = H \left\{ \tilde{\mathbf{u}} + \mu^2 \left[\left(Ah - \frac{H}{2}\right) (2\nabla h F_{22} + \nabla F_{21}) + \left(Bh^2 - \frac{H^2}{3}\right) \nabla F_{22} \right] \right.$$
$$+ \mu^4 \left[\left(Ah - \frac{H}{2}\right) (2\nabla h F_{42} + \nabla F_{41} + 2\nabla h F_{44} + \nabla F_{43}) \right.$$
$$+ \left(Bh^2 - \frac{H^2}{3}\right) (\nabla F_{42} + 3\nabla h F_{45} + \nabla F_{44})$$
$$\left.\left. + \left(Ch^3 - \frac{H^3}{4}\right) (4\nabla h F_{46} + \nabla F_{45}) + \left(Dh^4 - \frac{H^4}{5}\right) \nabla F_{46} \right] \right\} \tag{56}$$

where $H = h + \delta\eta$ denotes total water depth. The momentum equation may be written as

$$\mathbf{U}_t = -\nabla\eta - \frac{\delta}{2} \nabla \left(|\tilde{\mathbf{u}}|^2\right) + \Gamma_1 \left(\eta, \tilde{\mathbf{u}}_t\right) + \Gamma_2 \left(\eta, \tilde{\mathbf{u}}\right) \tag{57}$$

where

$$\mathbf{U} \;\equiv\; \tilde{\mathbf{u}} + \mu^2 \left[(A-1)\,h\,(2\nabla h F_{22} + \nabla F_{21}) + (B-1)\,h^2 \nabla F_{22}\right]$$

$$
\begin{aligned}
&+ \quad \mu^4\left[(A-1)\,h\,(2\nabla h F_{42} + \nabla F_{41} + 2\nabla h F_{44} + \nabla F_{43})\right.\\
&+ \quad (B-1)\,h^2\,(\nabla F_{42} + 3\nabla h F_{45} + \nabla F_{44})\\
&+ \quad \left.(C-1)\,h^3\,(4\nabla h F_{46} + \nabla F_{45}) + (D-1)\,h^4 \nabla F_{46}\right] \qquad (58)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_1 \;\equiv\;& \mu^2\nabla\left[\delta\eta F_{21t} + \left(2h\delta\eta + \delta^2\eta^2\right)F_{22t}\right]\\
&+ \quad \mu^4\nabla\left[\delta\eta\left(F_{41t} + F_{43t}\right) + \left(2h\delta\eta + \delta^2\eta^2\right)\left(F_{42t} + F_{44t}\right)\right.\\
&+ \quad \left(3h^2\delta\eta + 3h\delta^2\eta^2 + \delta^3\eta^3\right)F_{45t}\\
&+ \quad \left.\left(4h^3\delta\eta + 6h^2\delta^2\eta^2 + 4h\delta^3\eta^3 + \delta^4\eta^4\right)F_{46t}\right] \qquad (59)
\end{aligned}
$$

$$
\begin{aligned}
\Gamma_2 \equiv -\;& \mu^2\delta\nabla\left\{\tilde{\mathbf{u}}\cdot\left[(Ah - H)(\nabla F_{21} + 2\nabla h F_{22}) + \left(Bh^2 - H^2\right)\nabla F_{22}\right]\right.\\
&+ \quad \left.\frac{1}{2}(F_{21} + 2HF_{22})^2\right\}\\
&- \quad \mu^4\delta\nabla\left\{\tilde{\mathbf{u}}\cdot\left[(Ah - H)(\nabla F_{41} + 2\nabla h F_{42} + \nabla F_{43} + 2\nabla h F_{44})\right.\right.\\
&+ \quad \left(Bh^2 - H^2\right)(\nabla F_{42} + \nabla F_{44} + 3\nabla h F_{45})\\
&+ \quad \left.\left(Ch^3 - H^3\right)(\nabla F_{45} + 4\nabla h F_{46}) + \left(Dh^4 - H^4\right)\nabla F_{46}\right]\\
&+ \quad \frac{1}{2}\left|(Ah - H))(\nabla F_{21} + 2\nabla h F_{22}) + \left(Bh^2 - H^2\right)\nabla F_{42}\right|^2\\
&+ \quad \frac{1}{2}\left[(F_{21} + 2HF_{22})(F_{41} + 2HF_{42}\right.\\
&+ \quad \left.\left.F_{43} + 2HF_{44} + 3H^2F_{45} + 4H^3F_{46})\right]\right\} \qquad (60)
\end{aligned}
$$

Note that equation (57) retains the form of the Bernoulli term and thus is only applicable to irrotational motion. Uses of the model by Gobbi and Kirby (1999) were limited to one horizontal dimension, and thus no vertical vorticity is generated in any existing results.

The extension to $O(\mu^4)$ of the vertical structure of ϕ in the Gobbi et al. model provides a dramatic enhancement of the prediction of velocity components. For example, Figure 4 shows a comparison of vertical profiles of horizontal velocity for the linear, periodic wave case. The prediction is fairly robust up to $\mu = 5$, with a spurious flow reversal first occurring at $\mu = 5.54$ at a dimensionless elevation $z = -0.628$. Similar results are obtained for vertical velocities. Improvements are also documented in prediction of second-harmonic amplitudes over the lower order theory. Systematic improvements in leading order amplitude dispersion were, however, not noted.

Systematic improvements were also noted in solitary wave properties, including wave height, water particle velocity at crest, and total energy as functions of normalized phase speed. Figure 5 shows plots of a crest speed parameter ω_s given by

$$\omega_s = 1 - (u_c - F_r)^2 \qquad (61)$$

where u_c and F_r are horizontal particle velocity at the crest and wave phase speed both normalized by $\sqrt{gh}$. As the particle velocity varies from zero in linear waves to F_r at limiting wave height, ω_s varies

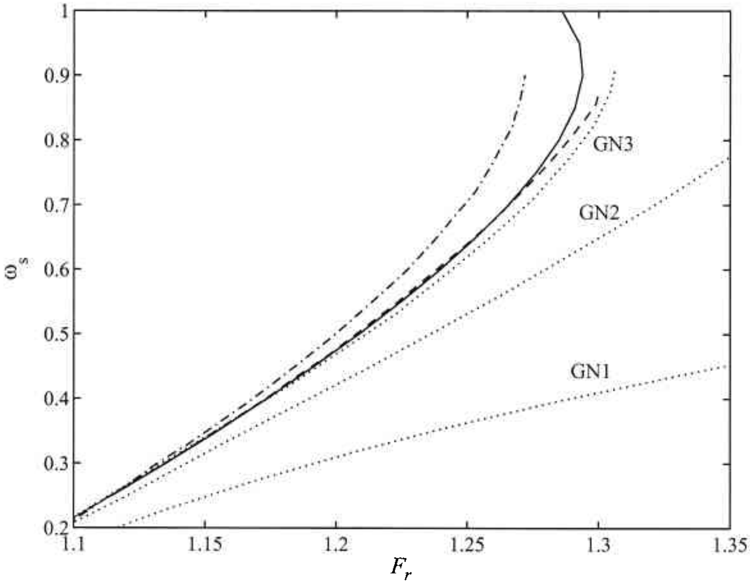

Figure 5. ω_s vs. phase speed for solitary waves. Exact (solid line), Gobbi et al. (dash), Wei et al. (dash-dot), Shields and Webster (dot). (From Gobbi et al., 2000).

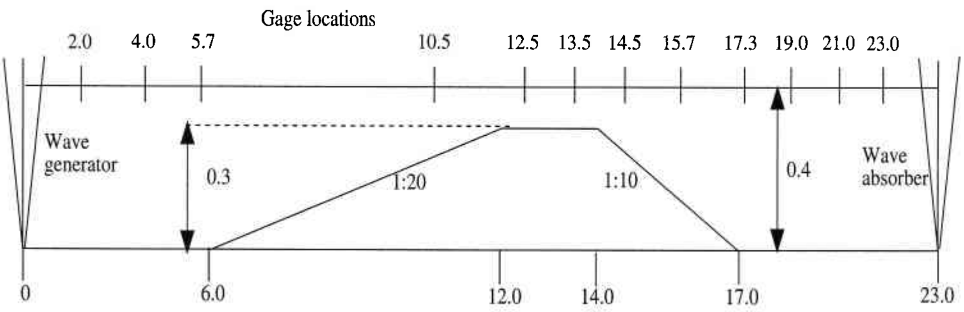

Figure 6. Sketch of wave flume in Delft experiments. All dimensions in (m). (From Gobbi and Kirby, 1999).

from zero to one. Figure 5 shows exact results due to Tanaka (1986), three levels of a Green-Naghdi theory due to Shields and Webster (1988), and results of the $O(\mu^2)$ theory of Wei et al. (1995) and the $O(\mu^4)$ theory of Gobbi et al. (2000). Results generally indicate that the present theory has accuracy comparable to level 3 Green-Naghdi theory, which has not been applied in realistic computational settings to date.

The most striking improvement in a practical sense is seen in a study of waves propagating from relatively deep water, over the shallow crest of a bar, and back into deep water, as presented originally in Beji and Battjes (1993). The test geometry and location of wave gauges is shown in Figure 6. Gobbi and Kirby considered two test cases for periodic wave propagation over the bar crest, using the theory of Gobbi and Kirby and the $O(\mu^2)$ theory of Wei et al. (1995). Figures 7 and 8 show results for Case (c), where waves with an initial $\mu = 1.69$ propagate from deeper water, over the shoal, and back into

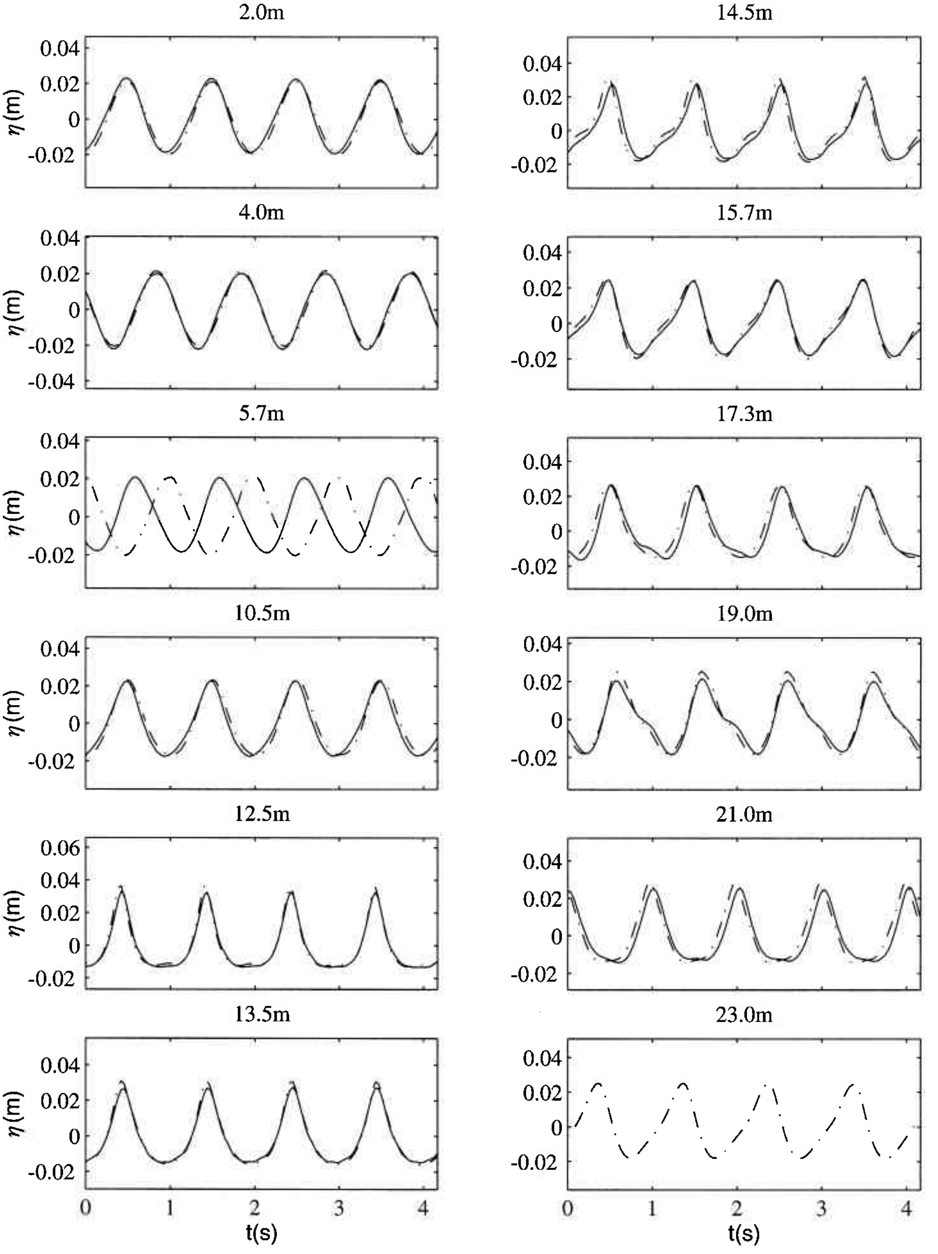

Figure 7. Comparisons of free surface displacement for case (c) of Delft experimental data at various gauge locations. Model of Gobbi and Kirby (dash-dot), data (solid). (From Gobbi and Kirby, 1999).

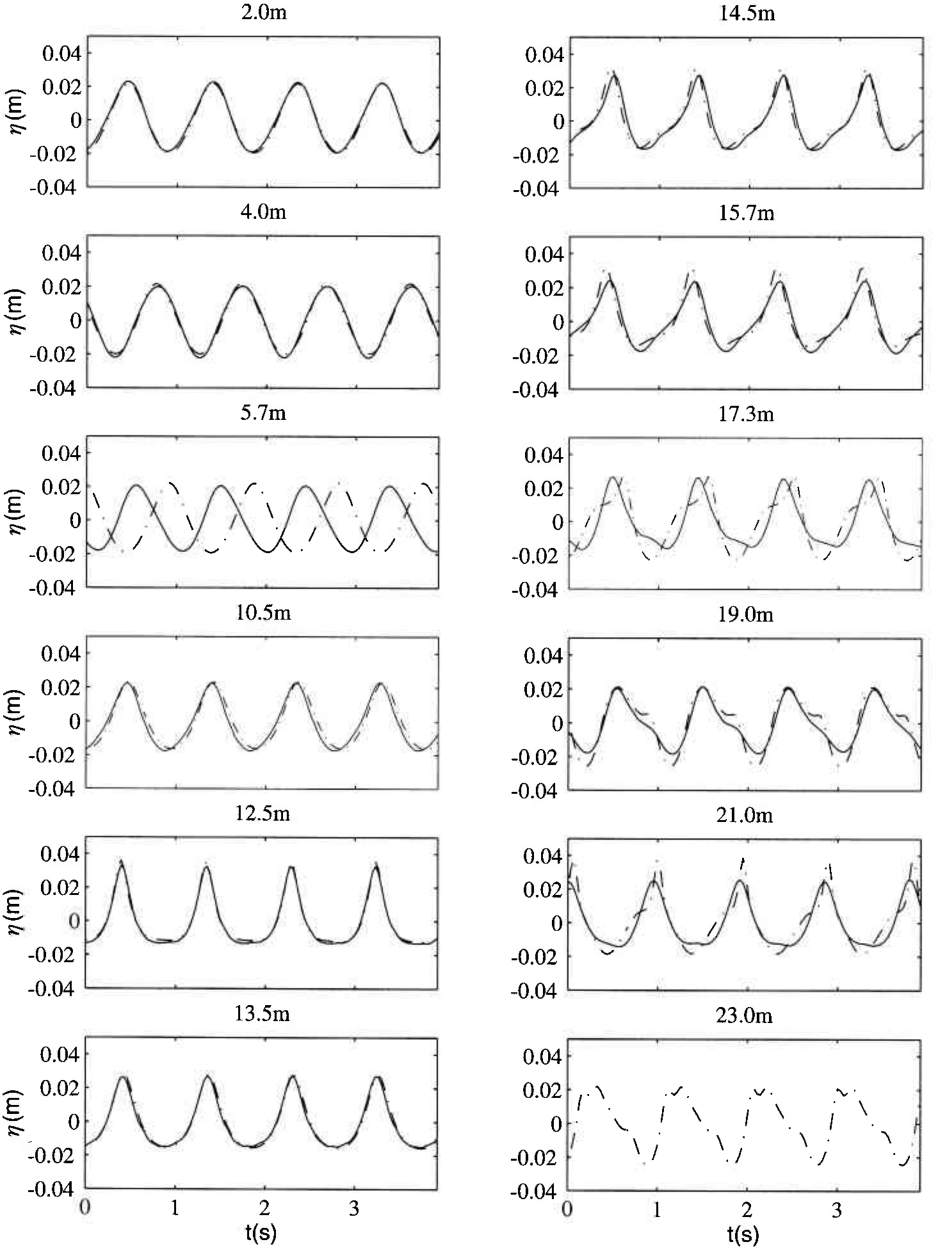

Figure 8. Comparisons of free surface displacement for case (c) of Delft experimental data at various gauge locations. Model of Wei et al. (dash-dot), data (solid). (From Gobbi and Kirby, 1999).

deeper water. Figure 7 shows results for the Gobbi and Kirby model, and indicates that the model is capable of preserving the overall shape of transmitted waves downstream of the bar, which is the most difficult region to obtain good results in. In contrast, Figure 8 shows results obtained using the Wei et al. model. In this case, there is clear damage to phase information in the transmitted wave, despite the fact that shoaling from deeper water to the bar crest was modeled relatively accurately. The accumulation of phase errors beyond the bar crest in the lower order model could be interpreted as being due to dispersion errors in the relatively less accurate model. However, this is not an entirely satisfactory explanation. In order to test this explanation, Gobbi and Kirby constructed a weakly nonlinear ($\delta = O(\mu^2)$) model system by truncating the Gobbi-Kirby model to $O(\mu^4, \delta\mu^2)$. Results from this case show a similar tendency to accumulate phase errors downwave of the shallow bar crest. Clearly, both full nonlinearity and enhanced dispersion effects play a crucial role in the accuracy of solutions in this case.

2.4.2. Model of Agnon et al. (1999) and Successors

An alternate approach to extending model derivation to higher order and increased accuracy was originally proposed by Agnon et al. (1999), with continuing development, for example, in Madsen et al. (2002). In this approach, the reduction of the problem to a description in terms of a velocity potential is dropped, and the identity of horizontal velocity $\mathbf{u}$ and vertical velocity w is retained instead. Following Madsen and Schäffer (1998), the irrotational solutions for the velocities may be written as

$$\mathbf{u}(\mathbf{x}, z, t) = \sum_{n=0}^{\infty}(-1)^n \left(\frac{z^{2n}}{(2n)!}\mu^{2n}\nabla(\nabla^{2n-2}(\nabla \cdot \mathbf{u_0})) + \frac{z^{2n+1}}{(2n+1)!}\mu^{2n+2}\nabla(\nabla^{2n} w_0) \right) \tag{62}$$

$$w(\mathbf{x}, z, t) = \sum_{n=0}^{\infty}(-1)^n \left(-\frac{z^{2n+1}}{(2n+1)!}\mu^{2n+2}\nabla^{2n}(\nabla \cdot \mathbf{u_0}) + \frac{z^{2n}}{(2n)!}\mu^{2n+2}\nabla^{2n} w_0 \right) \tag{63}$$

where, following equation (11),

$$\mathbf{u}_0 = \nabla\phi^{(0)}; \qquad w_0 = \phi^{(1)} \tag{64}$$

are the velocities at $z = 0$. The two velocities are related through the bottom boundary condition (equation (6)), giving a relation of the form

$$L_c\{w_0\} + L_s \cdot \{\mathbf{u}_0\} + \nabla h \cdot (L_c\{\mathbf{u}_0\} + L_s\{w_0\}) = 0 \tag{65}$$

where

$$L_c = \sum_{n=0}^{\infty}(-1)^n \frac{h^{2n}}{(2n)!}\nabla^{2n}, \qquad L_s = \sum_{n=0}^{\infty}(-1)^n \frac{h^{2n+1}}{(2n+1)!}\nabla^{2n+1} \tag{66}$$

A great deal of flexibility is left in choosing a procedure for improving the truncated series appearing in equation (65) during the development of a finite order theory. In Agnon et al. (1999), the series form of equation (65) is multiplied by a differential operator of the form

$$A = 1 + a_2 h^2\nabla^2 + a_4 h^4\nabla^4 + \cdots \tag{67}$$

and coefficients are then chosen to force terms to disappear in the resulting equation up to the required order. For example, for constant depth and a truncation level $N = 4$, the resulting equation is

$$(1 - \frac{4}{9}h^2\nabla^2 + \frac{1}{63}h^4\nabla^4)w_0 + (h\nabla - \frac{1}{9}h^3\nabla^3 + \frac{1}{945}h^5\nabla^5) \cdot \mathbf{u}_0 \tag{68}$$

which is correct to $O(\mu^8)$ and reproduces (4,4) Padé dispersion as in Gobbi et al. Agnon et al. extend this procedure to include mild-slope terms limited to $O(\nabla h)$, possibly limiting the accuracy of the resulting model when applied to abrupt nearshore bathymetry. Once the form of the relation between $\mathbf{u}_0$ and w_0 is obtained, Agnon et al. develop expressions for velocities at the free surface in terms of the velocities at $z = 0$, and then employ the evolution equations in terms of $\mathbf{V} = \nabla\Phi$, where $\Phi = \phi(\mathbf{x}, z = \eta, t)$ as in Dommermuth and Yue (1987) and others. Madsen et al. (2002) further generalize this procedure by choosing to develop series for the velocities at the arbitrary reference level z_α rather than the still water level $z = 0$. The additional freedom in the system of equations is then utilized to improve the vertical profile of horizontal velocity rather than the dispersion relation, which is already quite accurate.

The line of investigation initiated in these studies holds a great deal of promise. In particular, it is much more likely that any access to an actual model system for cases of higher accuracy than that in Gobbi et al. would come from this procedure rather than from direct expansion in terms of the velocity potential. Additional publications detailing both theoretical aspects and numerical treatments of the method will be forthcoming.

3. PRACTICAL EXTENSIONS TO THE $O(\mu^2)$ BOUSSINESQ MODEL

The $O(\mu^2)$ model written in terms of a velocity variable provides an easily extensible framework for the development of a nearshore processes model. In order to achieve this goal, various additional physical processes are incorporated, often on an ad hoc basis. In this section, we consider the extension of the model equation (22) to the form

$$\mathbf{u}_{\alpha t} + \delta(\mathbf{u}_\alpha \cdot \nabla)\mathbf{u}_\alpha + \nabla\eta + \mu^2\mathbf{V_1} + \delta\mu^2\mathbf{V_2} - \mathbf{R_b} - \mathbf{R_s} + \mathbf{R_f} = O(\mu^4) \tag{69}$$

where $\mathbf{R_b}$ denotes wave breaking effects, $\mathbf{R_f}$ denotes bottom friction, and $\mathbf{R_s}$ denotes subgrid-scale lateral mixing effects.

3.1. Improving Model Dispersion, Shoaling and Nonlinear Properties

As has been shown in a number of studies starting with Madsen et al. (1991), improvements in Boussinesq model performance can be introduced by employing rearrangements that alter the definitions of $\mathbf{M}$, $\mathbf{V_1}$ and $\mathbf{V_2}$ terms in the model equations, or by redefining the reference configuration of the dependent variables, leading to the same sort of rearrangement. The effectiveness of any modification can be established by examining linear properties such as shoaling and dispersion, along with nonlinear properties such as the amplitudes of harmonics or the magnitude of amplitude dispersion in Stokes wave solutions. We consider two cases in particular.

In order to provide a degree of freedom for use in optimizing nonlinear model properties, Kennedy et al. (2001) modified the formulation of Wei et al. (1995) by allowing the reference elevation z_α to depend on time and thus follow the rise and fall of the local water surface in some manner. The most general case considered defined z_α as

$$z_\alpha = \zeta h + \beta\delta\eta \tag{70}$$

where the original theory of Wei et al. is recovered by taking $\beta = 0$ and $\zeta = -1 + \sqrt{1/5}$, corresponding to the (2,2) Padé case. A special case of this relationship occurs when $\beta = \zeta + 1 = \sqrt{1/5}$, which gives

$$z_\alpha = -h + \beta H \tag{71}$$

This case corresponds to choosing a reference level which remains at a fixed proportion of the instantaneous total water depth, and would be equivalent to choosing a fixed σ_α reference level in a σ−coordinate model. The revision to the model equations is contained entirely within the $\mathbf{V_1}$ term (equation (24)), which is revised to read

$$\mathbf{V_1} = \left[\frac{1}{2} z_\alpha^2 \nabla(\nabla \cdot \mathbf{u}_\alpha) + z_\alpha \nabla(\nabla \cdot (h\mathbf{u}_\alpha))\right]_t - \nabla\left[\frac{1}{2}(\delta\eta)^2 \nabla \cdot \mathbf{u}_{\alpha t} + \delta\eta \nabla \cdot (h\mathbf{u}_{\alpha t})\right] \tag{72}$$

Kennedy et al. (2001) examined the optimal choice of β by examining a Stokes wave solution of the form

$$\eta = a_1[\cos(kx - \omega t) + a_+ \cos 2(kx - \omega t) + a_-] \tag{73}$$

where a_+ is a normalized second harmonic amplitude and a_- represents the steady set-down. Forcing the Taylor series expansions of the resulting second harmonic amplitude to match the Taylor series for the second harmonic amplitude of a regular Stokes wave for the full dispersive theory gives the choice $\beta = 17\sqrt{5}/200$. Figure 9 shows plots of the resulting second harmonic amplitudes and set-down, normalized by the correct solutions, for a range of $\mu = kh$ values. The second harmonic amplitude from the optimized model behaves quite well in comparison to the result from the original Wei et al. theory, and thus this revision to the original model is highly recommended.

Several studies have examined the enhancement of the $O(\mu^2)$ evolution equations in order to obtain higher-order dispersion. Schäffer and Madsen (1995) applied operators to rearrange dispersive terms in the weakly nonlinear model equations of Nwogu (1993) and obtained four sets of operator coefficients that would recover the more accurate (4,4) Padé dispersion relation in the model system, without adding higher-order terms as in the model of Gobbi et al. Madsen and Schäffer (1998)

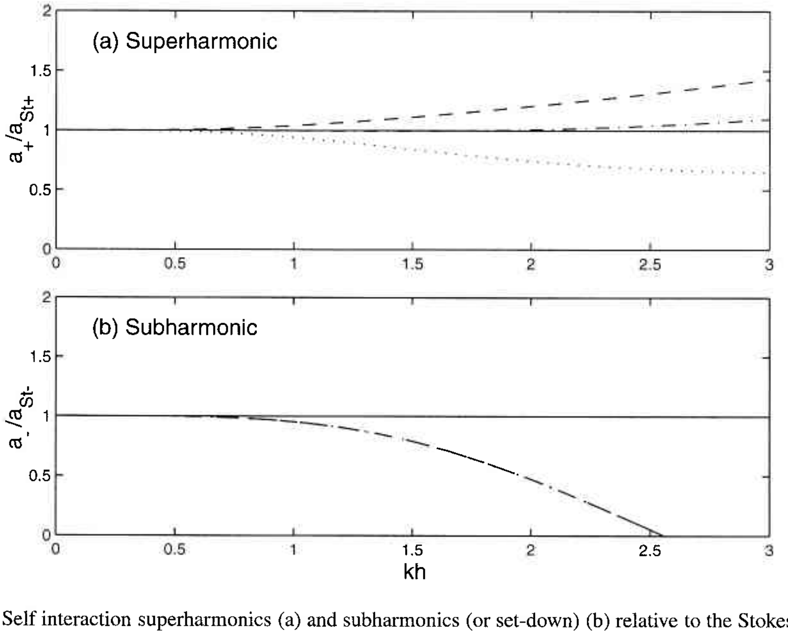

Figure 9. Self interaction superharmonics (a) and subharmonics (or set-down) (b) relative to the Stokes solution for the fully-dispersive problem. Original Wei et al. theory (dash), constant σ level theory (dot), optimized moving z_α theory (dash-dot). (From Kennedy et al., 2001).