Quantum Mechanics: Commutation Relation Proofs

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I. Proof for Non-Commutativity of Indivdual Quantum Angular Momentum Operators

In this section, we will show that the operators $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$ do not commute with one another, and hence cannot be known simultaneously. The relations are (reiterating from previous lectures):

$$\hat{L}_x = -i\hbar \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y} \right)$$

$$\hat{L}_y = -i\hbar \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z} \right)$$

$$\hat{L}_z = -i\hbar \left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x} \right)$$

We would like to proove the following commutation relations:

$$[\hat{L}_x, \hat{L}_y] = \mathrm{i}\,\hbar\,\hat{L}_z,$$

$$[\hat{L}_y, \hat{L}_z] = \mathrm{i}\,\hbar\,\hat{L}_x,$$

$$[\hat{L}_z, \hat{L}_x] = \mathrm{i}\,\hbar\,\hat{L}_y.$$

We will use the first relation for our proof; the second and third follow analogously. Let's also consider a function, $f(x, y, z)$ that we will have the operators act upon in our discussion. The expanded version of $[\hat{L}_x, \hat{L}_y] = \mathrm{i}\,\hbar\,\hat{L}_z$ is:

$$[\hat{L}_x, \hat{L}_y] = \left(\left(\hat{L}_x \right) \left(\hat{L}_y \right) - \left(\hat{L}_y \right) \left(\hat{L}_x \right) \right)$$

$$= \left(\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\right) - \left(\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\right)$$

Now let's expand the operators (ignore the underbraces and numbers for now):

$$\left(\hat{L}_x\right)\left(\hat{L}_y\right) = \left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)\left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)$$

$$= y\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial x}\right) - y\frac{\partial}{\partial z}\left(x\frac{\partial}{\partial z}\right) - z\frac{\partial}{\partial y}\left(z\frac{\partial}{\partial x}\right) + z\frac{\partial}{\partial y}\left(x\frac{\partial}{\partial z}\right)$$

$$= \underbrace{y\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\frac{\partial}{\partial x}\right)}_{\mathbf{1}} - \underbrace{y\left(x\frac{\partial}{\partial z}\frac{\partial}{\partial z}\right)}_{\mathbf{2}} - \underbrace{z\left(z\frac{\partial}{\partial y}\frac{\partial}{\partial x}\right)}_{\mathbf{3}} + \underbrace{z\left(x\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right)}_{\mathbf{4}}$$

Likewise for the $\left(\hat{L}_y\right)\left(\hat{L}_x\right)$ term:

$$\left(\hat{L}_y\right)\left(\hat{L}_x\right) = \left(z\frac{\partial}{\partial x} - x\frac{\partial}{\partial z}\right)\left(y\frac{\partial}{\partial z} - z\frac{\partial}{\partial y}\right)$$

$$= z\frac{\partial}{\partial x}\left(y\frac{\partial}{\partial z}\right) - z\frac{\partial}{\partial x}\left(z\frac{\partial}{\partial y}\right) - x\frac{\partial}{\partial z}\left(y\frac{\partial}{\partial z}\right) + x\frac{\partial}{\partial z}\left(z\frac{\partial}{\partial y}\right)$$

$$= \underbrace{z\left(y\frac{\partial}{\partial x}\frac{\partial}{\partial z}\right)}_{\mathbf{5}} - \underbrace{z\left(z\frac{\partial}{\partial x}\frac{\partial}{\partial y}\right)}_{\mathbf{6}} - \underbrace{x\left(y\frac{\partial}{\partial z}\frac{\partial}{\partial z}\right)}_{\mathbf{7}} + \underbrace{x\left(\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\frac{\partial}{\partial y}\right)}_{\mathbf{8}}$$

Combining terms and noting that $[\hat{x}, \hat{y}] = 0$ and $[\hat{p}_x, \hat{p}_y] = 0$. We will combine terms as follows: (1-8), (2-7), (3-6), and (4-5).

$$\begin{aligned}
(1-8) &= y\left(\frac{\partial}{\partial x} + z\frac{\partial}{\partial z}\frac{\partial}{\partial x}\right) - x\left(\frac{\partial}{\partial y} + z\frac{\partial}{\partial z}\frac{\partial}{\partial y}\right) \\
&= \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \left(yz\frac{\partial}{\partial z}\frac{\partial}{\partial x} - xz\frac{\partial}{\partial z}\frac{\partial}{\partial y}\right) \\
&= \left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + \left(zy\frac{\partial}{\partial x}\frac{\partial}{\partial z} - zx\frac{\partial}{\partial y}\frac{\partial}{\partial z}\right) \quad \textit{(recall the commutation relations noted!} \\
&= \underbrace{\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) + z\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right)\frac{\partial}{\partial z}}_{\mathbf{9}}
\end{aligned}$$

$$\begin{aligned}(2-7) &= xy\frac{\partial^2}{\partial z^2} - yx\frac{\partial^2}{\partial z^2} \\ &= (xy-yx)\,\frac{\partial^2}{\partial z^2} \\ &= [x,y]\,\frac{\partial^2}{\partial z^2} \\ &= (0)\frac{\partial^2}{\partial z^2} = 0\end{aligned}$$

$$\begin{aligned}(3-6) &= z^2\frac{\partial}{\partial x}\frac{\partial}{\partial y} - z^2\frac{\partial}{\partial y}\frac{\partial}{\partial x} \\ &= z^2\left(\frac{\partial}{\partial x}\frac{\partial}{\partial y} - \frac{\partial}{\partial y}\frac{\partial}{\partial x}\right) \\ &= z^2\,(\hat{p}_x\hat{p}_y - \hat{p}_y\hat{p}_x) \\ &= z^2\,[\hat{p}_x,\hat{p}_y] = 0 \qquad (\hat{p}_x \; and \; \hat{p}_y \; commute!)\end{aligned}$$

$$\begin{aligned}(4-5) &= zx\frac{\partial}{\partial y}\frac{\partial}{\partial z} - zy\frac{\partial}{\partial x}\frac{\partial}{\partial z} \\ &= \underbrace{z\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\frac{\partial}{\partial z}}_{\mathbf{10}}\end{aligned}$$

If we add the expressions 9 and 10, inserting a factor of $-i\hbar$ for each partial derivative that represents a momentum operator, we obtain:

$$\begin{aligned}[\hat{L}_x,\hat{L}_y] &= -\hbar^2\left(y\frac{\partial}{\partial x} - x\frac{\partial}{\partial y}\right) \\ &= \hbar^2\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)\end{aligned}$$

But,

$$\hat{L}_z = -i\hbar\left(x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}\right)$$

Thus,

$$[\hat{L}_x,\hat{L}_y] = \mathrm{i}\,\hbar\,\hat{L}_z$$