We said the approach to establishing probabilities for events is to
- Define the experiment
- List the sample points
- Assign probabilities to the sample points
- Determine the collection of sample points contained in the event of interest
- Sum the sample point probabilities to get the event probability

Ohio’s Learning, Earning and Parenting program (LEAP) is designed to encourage school attendance among pregnant and parenting teens on welfare.
- Eligible teens who provide evidence of school enrollment receive a bonus payment; those that don’t attend school have money deducted from their grant
- A survey was conducted of LEAP teens and the bonus/sanction requests were recorded for each teen

LEAP Example
- (N) No bonus or sanction 7%
- (OB) Only Bonus 37%
- (OS) Only sanctions 18%
- (BB) Both, but more bonuses 14%
- (BS) Both, but more sanctions 18%
- (BE) Both, equal 6%
  100%

Venn Diagram of LEAP

S is the Event involving Sanctions
B is the event involving Bonus
What is the probability that both bonuses and sanctions are requested for a LEAP teen?

This can be thought of as event C (for both)

Or the intersection between A and B

And we sum the probabilities for

\[ P(BB) + P(BS) + P(BE) = \]

\[ .14 + .18 + .06 = .38 \]

What is the probability of Sanctions or Bonuses?

An alternative way to say this is the compliment, or

1 - the probability of neither

Neither is N, or No bonus or sanction

\[ P(N) = .07 \]

\[ 1 - .07 = .93 \]

A traditional way to view this is:

\[ P(S) + P(B) - P(S \cap B) \]

\[ P(S) = OS + BB + BS + BE \]

\[ = .18 + .14 + .18 + .06 = .56 \]

\[ P(B) = OB + BB + BS + BE \]

\[ = .37 + .14 + .18 + .06 = .75 \]

\[ P(S \cap B) = BB + BS + BE \]

\[ = .14 + .18 + .06 = .38 \]

\[ = .56 + .75 - .38 = .93 \]

The complement of an event \( A \) is the event that \( A \) does not occur - that is all sample points not in Event \( A \)

Denoted as \( A^c \)

\[ \text{The } P(A) + P(A^c) = 1.0 \]
Compound Events

Events can be comprised of several events joined together, and these are called **COMPOUND EVENTS**. They can be the UNION of several events or the INTERSECTION of several events.

Union of two Events p. 111

- The union of two events, A and B is the Event that occurs if either A, B, or both occur on a single performance of the experiment.
- We denote the union as $A \cup B$.
- $A \cup B$ consists of all the sample points that belong to A or B or both.

Intersection of two Events p. 111

- The intersection of two events, A and B is the Event that occurs if both A and B occur on a single performance of the experiment.
- We denote the intersection as $A \cap B$.
- $A \cap B$ consists of all the sample points that belong to both A and B.

Example using a die toss

- Event A [Toss an even number]
- Event B [Toss a number <= 3]
- What is $A \cap B$?
- What is $A \cup B$?
- Can you calculate the probability of the union and the intersection of these events?

What is $A \cap B$ for a roll of a die?

- $A = \{2, 4, 6\}$
- $B = \{1, 2, 3\}$
- $A \cap B = \{2\}$

What is $A \cup B$ for a roll of a die?

- $A \cup B$ consists of all the sample points that belong to either A or B.
What is $P(A \cap B)$ for a roll of a die?

![Venn Diagram]

The probability of this event is

$$P(A \cap B) = P(1) + P(2) + P(3) + P(4) + P(6)$$

Another way to approach the problem of $A \cap B$

- Find the probability of the compliment, and subtract from 1
  - $1 - P(A \cap B)$
- This would mean everything that wasn’t in events A or B
  - In this case it is the value of 5
  - And the probability of rolling a five is $1/6$
  - $1 - 1/6 = 5/6$

What is $A \cup B$ for a roll of a die?

- $A = \{2, 4, 6\}$
- $B = \{1, 2, 3\}$
- $A \cup B =$

The probability of this event is

$$P(A \cup B) = P(2)$$

Additive Rule of Probability

- $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
- This is called the Additive Rule of Probability
- And if events A and B are mutually exclusive, meaning no intersection, then
  - $P(A \cup B) = P(A) + P(B)$

Roulette example, p 120

- Roulette is a betting game where a ball spins on a circular wheel that is divided into 38 arcs of equal length
- **Red Numbers**
  - 1 2 3 4 5 6 7 8 9 10 11 12 13 14 15 16 17 18 19 20 21 22 23 24 25 26 27 28 29 30 31 32 33 34
- **Black numbers**
  - 2 4 6 8 10 11 13 15 17 20 22 24 26 28 30 31 32 33 34 35
- **Green numbers**
  - 00 0
- You can bet on odd, even, red, black, high, low
Roulette example
- A: Outcome is an odd number (note 00 and 0 are neither even or odd)
  - A: [1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35]
  - P(A) = 18/38 = .474

Roulette example
- B: Outcome is a black number
  - B: [2, 4, 6, 8, 10, 11, 13, 15, 17, 20, 22, 24, 26, 28, 29, 31, 33, 35]
  - P(B) = 18/38 = .474
- C: Outcome is a low number (1-18)
  - C: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]
  - P(C) = 18/38 = .474

Find the Intersection of Events A and B
- Find all numbers that are both odd and black
- A: [1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35]
- B: [2, 4, 6, 8, 10, 11, 13, 15, 17, 20, 22, 24, 26, 28, 29, 31, 33, 35]
  - A \cap B = \quad P(A \cap B) =

Find the Intersection of Events A and C
- Find all numbers that are both odd and low
- A: [1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35]
- C: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]
  - A \cap C = \quad P(A \cap C) =

Find the Intersection of Events B and C
- Find all numbers that are both black and low
- B: [2, 4, 6, 8, 10, 11, 13, 15, 17, 20, 22, 24, 26, 28, 29, 31, 33, 35]
- C: [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18]
  - B \cap C = \quad P(B \cap C) =

What about the Intersection between all three Events
- Find all odd numbers that are Black and are low
- A B C = \quad P(A \cap B \cap C) = 4/38 = .105
What about the union between two events? A $\cap$ B
This is all the numbers that are odd or are black.
Note: there is overlap between them, and they are not mutually exclusive.

All odd numbers and all that are Black:
1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 2, 4, 6, 8, 10, 20, 22, 24, 26, 28.
The green numbers (also in bold) represent the overlap.
P(A $\cap$ B) =

Use the additive rule:
P(A $\cap$ B) = P(A) + P(B) - P(A $\cap$ B)
P(A $\cap$ B) = .474 + .474 - .211
P(A $\cap$ B) = .737

All points that are either odd, Black, or low:
1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31, 33, 35, 2, 4, 6, 8, 10, 20, 22, 24, 26, 28, 12, 14, 16, 18.
32/38 = .842

So what’s the point??
Not to make you better gamblers.
I do want you to understand:
that probability is based on events within a sample space.
that defining the probability of an event(s) can be complicated, particularly for compound events.
That we will use probability theory when to make inferences.
Conditional Probability
- If we have knowledge that affects the outcome of an experiment, the probabilities will be altered
- We call this a **Conditional Probability**
- The probability of an event is conditioned on another event
- We often use the term “given” when talking about conditional probabilities

Designated as \( P(A|B) \)

Suppose we have the roll of a die
- \( P(\text{even number}) = P\{2, 4, 6\} = \frac{3}{6} = .5 \)

What if we ask the probability of an even number given the die is less than or equal to 3?
- \( P(\text{even} | \#3) = P\{2 | \#3\} = \frac{1}{3} \)

Note: it is a 2 out the new or given possible space \{1,2,3\}

The formula of a conditional probability is:
- Probability of the intersection of A and B
- Divided by the probability of B
- It adjusts the probability of the intersection to the reduced sample space of the condition

\[
P(A | B) = \frac{P(A \cap B)}{P(B)}
\]

Back to the die example
- If \( A = \{\text{even number on a die}\} \)
- \( B = \{\text{less than or equal to 3}\} \)
- \( P(A \cap B) = P(2) = \frac{1}{6} = .1667 \)
- \( P(B) = P(1) + P(2) + P(3) = \frac{3}{6} = .5 \)
- \( P(A|B) = .1667/ .5 = .333 \)
- or \( 1/3 \)

Multiplicative Rule
- The multiplicative rule shows us the probability of an intersection
- Remember we said a conditional probability is determined by the formula

\[
P(\text{intersection}) = \frac{P(A \cap B)}{P(B)}
\]

This shows the probability of an intersection
- It suggests that the probability of an intersection between two events depends upon the conditional probability between the two events

\[
P(A \cap B) = P(B)P(A | B)
\]
### Probability of an Intersection

\[ P(A \cap B) = P(B)P(A \mid B) \]

\[ P(B \cap A) = P(A)P(B \mid A) \]

### Multiplicative Rule and Independence

- If Events A and B are independent of each other
  - Then, \( P(A \mid B) = P(A) \)
  - Independence means that the probability of A doesn’t change given the event B
  - And likewise, \( P(B \mid A) = P(B) \)

### Multiplicative Rule and Independence

- In the case of independence between events A and B
  - The formula for probability of an interaction reduces to:
    \[ P(A \cap B) = P(A)P(B) \]
  - If we can assume independence between events, figuring the probability of the intersection of events is much easier

### Look at the Probability of tossing two coins

What is the probability of getting two heads?

First Toss
- .5
  - H
  - T

Second Toss
- .5
  - H
  - T

- H H = .25
- H T = .25
- T H = .25
- T T = .25

### Earlier we looked at this problem differently

- Sample space for flipping two coins and noting the face:
  - Observe H H \( \frac{1}{4} = .25 \)
  - Observe H T \( \frac{1}{4} = .25 \)
  - Observe T H \( \frac{1}{4} = .25 \)
  - Observe T T \( \frac{1}{4} = .25 \)

### Use Multiplicative Rule and Independence

- The probability of observing a head in a single flip of a coin is \( \frac{1}{2} \) or .5
- The probability of observing a tail in a single flip of a coin is \( \frac{1}{2} \) or .5
- If I can assume independence
  - \( P(\text{two Heads}) = \)
  - \( = P(\text{Head 1st flip}) @ P(\text{Head on the 2nd flip}) \)
  - \( = (.5) @ (.5) = .25 \)
Multiple through to get the probabilities

What is the probability of getting two heads?

```
First Toss
.5
H
.5
H
1
TT
HH
TT
HHH
HT
THT
TTH
TTT
HH
First
Toss
Second Toss
.5
H
.5
T
.5
H
.5
T
.5
T
.5
T
HHHH
HTHT
THTH
TTTT
```

Conditional Probability

Conditional probability and independence are very important concepts in research.
- If we hypothesize salary levels differ between men and women, in essence we are saying, "given you are a female, I expect your salary is lower."
- If we hypothesize that level of response is different between a drug and the treatment group, we are saying, "given you received the drug your response is higher."
- Relationships that show independence demonstrate there is no effect or change.

A little circular?

**Probability of A Union**

\[ P(A \cup B) = P(A) + P(B) - P(A \cap B) \]

**Conditional Probability**

\[ P(A \mid B) = \frac{P(A \cap B)}{P(B)} \]

**Probability of an Intersection**

\[ P(A \cap B) = P(B)P(A \mid B) \]

Example

- Deck of Cards
- Four suits, Hearts, Diamonds, Clubs, Spades
- 13 in each suit
- Let Event A = Face Cards (Jack, Queen, King)
- Let Event B = Hearts
- Let Event C = Red suit (Hearts, Diamonds)

Define Experiment: selecting from a single Deck of cards

List the sample points

- Event A = [J ♥ Q ♥ K ♥; J ♦ Q ♦ K ♦; J ♣ Q ♣ K ♣]
- Event B = [A ♥ 2 ♥ 3 ♥ 4 ♥ 5 ♥ 6 ♥ 7 ♥ 8 ♥ 9 ♥ 10 ♥ J ♥ Q ♥ K ♥]
- Event C = [A ♥ 2 ♥ 3 ♥ 4 ♥ 5 ♥ 6 ♥ 7 ♥ 8 ♥ 9 ♥ 10 ♥ J ♥ Q ♥ K ♥; A ♦ 2 ♥ 3 ♥ 4 ♥ 5 ♥ 6 ♥ 7 ♥ 8 ♥ 9 ♥ 10 ♥ J ♥ Q ♥ K ♥]

Assign Probabilities

- A random draw of each card is 1/52
- \( P(\text{Event A}) = \frac{9}{52} = .1731 \)
- \( P(\text{Event B}) = \frac{13}{52} = .25 \)
- \( P(\text{Event C}) = \frac{26}{52} = .5 \)
What is the Probability of:
- $P(A \cap B)$
- $P(B \cap C)$

What is the Probability of:
- $P(A \cup B)$
- $P(B \cup C)$

What is the Probability of:
- $P(A|B)$