Statistics Program – Food and Resource Economics
University of Delaware – Fall 2006

STAT 601 Final Exam (Eggermont) Name: ____________

• Do 8 problems (out of 9).

1. Let $B \sim \text{Bernoulli}(\frac{1}{2})$ and $Y \mid B \sim \text{Uniform}(B-1, B)$.
   (a) Show that $Y$ and $B$ are uncorrelated.
   (b) Show that $Y$ and $B$ are not independent.
   (c) Compute the moment generating function of the bivariate random variable $(Y - B, B)$.
   (d) Does your answer to (c) help when deciding whether $Y - B$ and $B$ are independent?

2. Let $X, Y, Z$ be independent and normally distributed random variables with equal variances but possibly unequal means,
   
   $X \sim \text{Normal}(\mu_X, \sigma^2), \quad Y \sim \text{Normal}(\mu_Y, \sigma^2), \quad Z \sim \text{Normal}(\mu_Z, \sigma^2)$.

   Show that $U, V$ and $W$ are (jointly) independent.

3. Let $U_1, U_2 \sim \text{Uniform}(0, 1)$ be independent. Determine the distribution of $|U_1 - U_2|$.

4. Let $U_1, U_2, \cdots, U_n$ be independent Uniform(0, 1) random variables. Show that
   \[ \sqrt{n} \min_{1 \leq i \leq n} U_i \longrightarrow 0 \quad \text{in probability}. \]

5. For $n \in \mathbb{N}$, let $N \sim \text{Poisson}(n)$.
   (a) Compute the moment generating function of $N$.
   (b) Compute the moment generating function of $(N - n)/\sqrt{n}$.
   (c) Show that
   \[ \frac{N - n}{\sqrt{n}} \longrightarrow Z \sim \text{Normal}(0, 1) \quad \text{in distribution}. \]
6. (Parts (a) and (b) are not related; they are not unrelated either.)

(a) Let $\bar{X}_1$ and $\bar{X}_2$ be the (sample) means of two independent samples of a $\text{Normal}(\mu, \sigma^2)$ random variable, of sample sizes $n$ and $m$, respectively. Let $0 < \alpha < 1$ (think $\alpha = 0.05$) and let $z_\alpha$ denote the critical value at the $\alpha$ level, i.e., such that

$$\mathbb{P}[Z > z_\alpha] = \alpha,$$

where $Z \sim \text{Normal}(0, 1)$. Determine the quantity $d_\alpha$ (in terms of the variables $n, m, \mu, \sigma^2$) such that

$$\mathbb{P}[\bar{X}_1 - \bar{X}_2 > d_\alpha] = \alpha.$$

(b) A one-gallon container of Neapolitan ice cream contains vanilla, strawberry and chocolate ice cream. The amount of each is (approximately) a normally distributed random variable, with mean $\frac{1}{3}$ gallon and standard deviation $0.01$. Assume that the amounts of each kind of ice cream are independent (within limits, this seems reasonable). Compute (an expression for) the probability that a randomly picked container contains more than 1.03 gallon of ice cream.

7. Let $X_1, X_2, \ldots, X_1$ be a random sample of a $\text{Gamma}(\alpha, \beta)$ random variable. Let

$$Y = \sqrt{n}(\bar{X} - \alpha \beta).$$

(a) Compute the moment generating function, $M_Y(t)$, of $Y$.

(b) Evaluate

$$\lim_{n \to \infty} M_Y(t).$$

[“Evaluate” is not the same as “Compute”.]

8. Let $U_1, U_2, U_3 \sim \text{Uniform}(0, 1)$ be independent, and let $U_{(1)} < U_{(2)} < U_{(3)}$ denote the order statistics. Show that

$$R_1 \overset{\text{def}}{=} \frac{U_{(1)}}{U_{(2)}}, \quad R_2 \overset{\text{def}}{=} \frac{U_{(2)}}{U_{(3)}}, \quad \text{and} \quad U_{(3)}.$$
are independent. [Hint: Choose your own poison, but you might want to consider computing $\mathbb{P}[R_1 < t \& R_2 < s \& U_{(3)} < r]$.]

9. Consider the following way of randomly breaking a stick in three pieces. Assume the length of the stick equals 1. Let

$$U_1, U_2, U_3 \sim \text{Uniform}(0,1)$$

be independent, and define the lengths of the three pieces as

$$\ell_i = \frac{U_i}{U_1 + U_2 + U_3}, \quad i = 1, 2, 3.$$

Now break the stick into three pieces with the above lengths (they do indeed add up to one). We want to make a triangle out of the three pieces. One verifies that this is possible if and only if

the longest piece is shorter than the sum of the other two pieces.

(a) Find a concise expression for the probability that one can make a triangle out of the three pieces. [Hint: Order statistics !!!]

(b) Write your answer under (a) as a repeated integral (pay attention to integration bounds).

(c) Evaluate the expression under (b). [This may be tricky and time-consuming!]